Astron. Observ.

THE MATHEMATICAL THEORY

OF

ECLIPSES

ACCORDING TO CHAUVENET'S TRANSFORMATION OF

BESSEL'S METHOD

EXPLAINED AND ILLUSTRATED

TO WHICH ARE APPENDED

TRANSITS OF MERCURY AND VENUS AND OCCULTATIONS OF FIXED STARS

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PREFACE.

THE present work is designed as a convenient hand-book for the computer of Solar and Lunar Eclipses. It has originated partly from a manuscript book of formulæ and precepts, prepared by the author for his own use, during the period of twenty-four years, in which, besides other astronomical work, he has been engaged in computing the eclipses for the *American Ephemeris and Nautical Almanac.* The publication was suggested by reason of sundry applications to him, both orally and by letter, from persons desiring information on the subject of eclipses, and also explanations of difficult passages in CHAUVENET'S *Astronomy.*

The chapter on Eclipses in this work—*Spherical and Practical Astronomy,* by Professor WILLIAM CHAUVENET, published in 1863—is the most thorough and exhaustive treatise on this subject that has yet been published. It is taken as the standard authority for the methods of computation by both the English and the American *Nautical Almanacs.*

The theory is one of considerable intricacy, and while the student can generally follow CHAUVENET in deducing the formulæ, yet he meets with some difficulty in grasping the subject, finding many points requiring further explanation than is given in the text. No other publication is of the least assistance in explaining CHAUVENET'S chapter.

There is another difficulty encountered by the practical computer, no matter how well skilled he may be in mathematical formulæ or the art of computing; and that is, the formulæ in CHAUVENET are not given in the order they are to be used, but just as they chance to be derived. They are so numerous, and so much scattered, that it is almost imperatively necessary for the computer to write them off in the order they are to be taken up. This has been done in the present work. Moreover, some of the formulæ given by CHAUVENET, adding to the completeness of his work, are not absolutely necessary, and may be disregarded. These also have been here pointed out.

And, further, CHAUVENET'S admirable treatise is so rigorously exact in all its formulæ, that many of them, especially those required only for projection on a chart, may be considerably simplified, and the labor of using them thereby diminished. This has also been pointed out in the proper places.

Finally, during the time the author has been engaged upon the *Nautical Almanac* in computing the eclipses (which is longer than that of any of his predecessors upon this portion of the *Almanac*), the eclipses have rotated through somewhat more than one Saros of eighteen years, after which they repeat themselves very similarly. Their peculiarities have been here noted, together with many details not before published in any astronomy; difficult passages in CHAUVENET'S chapter, so far as they have been called to the author's attention, are here explained; some errata and other mistakes in the text and in one figure, noticed; and the formulæ themselves explained graphically, with numerous precepts given for the practical computer.

The graphic method here employed for explaining the formulæ is a feature of this work, which the author believes has not heretofore been made use of so extensively for explaining a complicated series of formulæ. The eclipse is dissected after the manner of a surgeon —it is cut up and the hidden parts laid open to view.

The whole subject is treated pretty much as a lecturer might address his class after closing his text-book.

The author desires to express his obligations especially to LOUIS CHAUVENET, Esq., of St. Louis, son of Professor CHAUVENET, and holder of the copyright of his *Astronomy*, for his cordial and full permission to quote from his father's work; and also to Professor WALTER S. HARSHMAN, U. S. N., Director of the *Nautical Almanac* office, for the privilege of photographing the author's original drawings made for that office, of the solar eclipses of September 9, 1904, and August 8, 1896, which are reproduced in the present work. Also to Professor ASAPH HALL, U. S. N.; to Professor FRANK H. BIGELOW, of the Weather Bureau, and to Mr. J. ROBERTSON, of the *Nautical Almanac* office, for reading portions of the manuscript of this work.

THE AUTHOR.

2015 Q STREET, WASHINGTON, D. C.
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ANALYTICAL TABLE OF CONTENTS.

Part I.—SOLAR ECLIPSES.

PART II.

PRINCIPAL NOTATION.

[Other quantities not generally used are defined on the pages where they occur.]

	☉	The sun.
	☾	The moon.
	✳	A fixed star, Section XXVI., Occultations.
	☌	Conjunction, at which time Solar eclipses occur, Arts. 12, 21.
	☍	Opposition, at which time Lunar eclipses occur, Art. 190.
	☊	Ascending node of an orbit; chiefly here, that of the moon, Art. 5.
	:	Sign of Division in the Examples.
*	$l'\ l$	Sun and moon's longitudes, Art. 5.
*	β	Moon's latitude, Art. 5.
	$\alpha'\ \delta'\ r'$ $\alpha\ \delta\ r$	Sun's Moon's } R. A., Dec., and distances from the earth, Arts. 15, 17.
	$a\ d$	Right ascension and declination of the point Z, Arts. 25, 38.
*	d_1	Transformation of d to take account of the earth's spheroid, Arts. 32, 42.
	π_0	Sun's mean horizontal parallax, Arts. 15, 21.
	$\pi'\ \pi$	Sun and moon's equatorial horizontal parallaxes, Art. 7.
	π_1	Moon's parallax reduced to latitude 45° (Lunar Eclipses), Art. 189.
	H	Sun's mean semidiameter, Art. 21.
	$s'\ s$	Sun and moon's true semidiameters, Art. 21.
	k	Ratio of moon's semidiameter s, to that of the earth π, Section XXI.
	G	Distance between the sun and moon, Art. 25.
*	b	Ratio of moon's distance r to that of the sun r', Art. 25.
	g	$=1-b$, Art. 25.
	I	Inclination of the moon's orbit to the ecliptic, Art. 8.
	$\Theta\ \mu$	Sidereal Time, Art. 27.
	μ_1	Hour angle of the point Z at the Greenwich Meridian, Art. 27.
	μ'	Its hourly variation.
*	h'	Mean time hours reduced to sidereal time, Art. 27.
	$x\ y\ z$	Coördinates of the centre of the shadow, Art. 28.
	$x_0'\ y_0'$	Mean hourly variations, Art. 30.
	$x'\ y'$	Absolute hourly variations, Art. 30.
	y_1	y transformed to take account of the earth's spheroid, Art. 33.
*	$l\ l_1$	Radii of the penumbra and umbra on fundamental plane, Art. 31.
	L	Radius on the earth's surface, Arts. 116, 150.
	L	Also radius of the earth's shadow in Lunar eclipses, Art. 189.
	f	$=\mu' \cos d$. Quantities in the Eclipse Tables, Arts. 27, 34.
	$f\ f_1$	Angles of the cones of shadow, Art. 31.
	$i\ i_1$	$i=\tan f$, $i_1=\tan f_1$ as used in the formulæ, Art. 31.
	$c\ c_1$	Distances of the vertices of cones above fundamental plane, Art. 31.
	$C\ C_1$	Constants for cones, Art. 31.
	$C\ c$	Auxiliaries in the General Formulæ, Arts. 81, 86.

* These symbols or others like them are found in other parts of this list.

*	$b' \; c'$	Quantities in Eclipse Tables, Art. 32.
	$b'' \; c''$	Their hourly variations, Art. 32.
	p	Radius of the earth in the fundamental plane, Arts. 50.
	ρ	The earth's radius for any place, Arts. 42, 149.
	ρ_1	Transformation of ρ, Arts. 32, 42.
	E	Angle of the resultant of the motion of the shadow and a point on the earth's surface, Arts. 32, 44, 70, 97.
	e	The distance at each hour, Art. 32, etc.
	E	Equation of Time (chiefly used in Lunar Eclipse), Arts. 163, 189.
	e	Eccentricity of the earth's spheroid, Art. 20.
*	M	Magnitude of an eclipse, Solar, Arts. 79, 156; Lunar, Art. 191.
	M	Angle of positions of any point of centre line with axis Y, Arts. 58, 79.
	m	Its distance on the fundamental plane, Arts. 58, 79.
	Z	Zenith, The axis of Z, Pole of fundamental plane, Art. 38.
	N	Angle of the path of shadow with axis of Y, Arts. 95–7.
	n	Motion of shadow in one hour, Arts. 95–7.
	Q	Angle of position of centre of shadow from any point on the cone of shadow, Arts. 80, 94, 97, 118, 137.
	ϕ	Geographical latitude of any place, Art. 149.
	ϕ'	Geocentric latitude of the same place, Art. 149.
	ϕ_1	Latitude to take account of the earth's spheroid, Art. 42.
	$\xi \; \eta \; \zeta$	Coördinates of any place on the earth's surface, Art. 85.
	$\xi' \; \eta' \; \zeta'$	Their hourly variations.
	ζ	Sun's zenith distance, Section XX.
	β	The coördinate $\zeta_1 \; \zeta_1 = \cos \beta$, Arts. 81, 85.
	$\eta_1 \; \zeta_1$	Transformation for the earth's spheroid, Art. 85.
	$\gamma \; \gamma'$	Angles of position on fundamental plane, Art. 50.
	ψ	Angle in the Extreme times explained in Art. 58. Another angle in Maximum curve defined, Arts. 72 to 75.
	θ	Parallactic angle, Art. 87.
*	h	Altitude of the sun above the horizon, Art. 87. Height of a place above fundamental plane, Art. 87.
	ϑ	Hour angle of any place, and Local Apparent Time Arts. 50, 108, 109, 157.
	ω	West Longitude of any place $\omega = \mu_1 - \vartheta$, Art. 50.
	T_0	Epoch Hour, Art. 30, or assumed time.
	τ	Computed correction for assumed time T_0.
	T	The time of any phenomenon $T = T_0 + \tau$.
*	$M \; D$	Magnitude or Degree of obscuration, Arts. 79, 189, 191.
	Σ	Sum of any two or more quantities, Art. 30.
	Δ	Used to denote a distance, Art. 68.
	Δ or d	Finite differences of computed quantities, Arts. 18, 29.
*	d	Symbol of differentiation, Arts. 28, 182.
	V	Angle of position from vertex of sun or moon, Art. 153.
	$l - l$	Notation in Addition and Subtraction logarithms, Arts. 28, 35.

Constants used in Eclipses, see Arts. 20, 23.

LIST OF PLATES AND FIGURES.

WITH INDEX TO GEOMETRICAL CONSTRUCTIONS.

THE

THEORY OF ECLIPSES.

SECTION I.

INTRODUCTION.

ARTICLE 1.—*Retrospect.*—There are two methods, radically different from each other, of computing a solar eclipse. That known to the earlier astronomers consists in finding the times when the disks of the sun and moon are tangent in a visual line from the observer; or, in other words, when the centres of the sun and moon are distant from one another by an arc in the celestial sphere equal to the sum of their semidiameters.

It was not until the time of BESSEL and HANSEN in the early part of the nineteenth century that this subject was fully developed, and another method devised for computing solar eclipses, by considering the cone of shadow cast by the moon, and its passage over the surface of the earth; and similarly for lunar eclipses the passage of the moon through the earth's shadow.

In solar eclipses, the first method is restricted to but *one* place on the earth's surface at a time; given the *place*, the *times* may be found. The second method is the reverse of this; given the *time*, and the *place* where the phenomenon is seen may be found. Thence by assuming a series of times, all places on the earth's surface where the *same* phenomena are visible may be made known, an advantage which the first method does not possess. The position of the cone of shadow is determined by its axis,—the line joining the centres of the sun and moon,—and all points of the earth and shadow are referred to a *Fundamental Plane* taken at right angles to the axis, and passing through the centre of the earth. A set of coördinate axes may now be assumed at pleasure. BESSEL, who wrote in 1841–42,* by a happy thought selected the intersection of this

* *Astronomische Untersuchungen*, 2 vols., 1841–42.

plane with the earth's equator as the axis of X. HANSEN, who followed him in 1858,* assumed the intersection of this plane with the ecliptic as the axis of X. The former method has many advantages over the latter. "As a refined and exhaustive disquisition upon the whole theory," says Professor CHAUVENET,† "BESSEL'S *Analyse der Finsternisse*, in his *Astronomische Untersuchungen*, stands alone."

Professor CHAUVENET has generally followed BESSEL'S method; but in many of the problems has given his own solutions, making his work the best yet published on the subject of eclipses.

While Astronomy was advancing in this direction, other branches of science were no less active in lines that were destined eventually to have great influence upon Astronomy. As early as 1814, though they had been observed previously, FRAUNHOFER studied the dark lines in the solar spectrum to which his name has been given; but it was not until 1859, when KIRCHOFF made his great discovery of the reversal of the spectrum, that their nature and origin became known, and Spectrum Analysis burst into a science, the connecting link between Chemistry and Astronomy.

In the arts also Photography was progressing, and in 1860 was first made use of in observing a solar eclipse. It was at once seen that during a total eclipse spectrum analysis could advantageously be used for observation, but it was not until 1868 and 1869 that an opportunity occurred for this purpose. Total solar eclipses then assumed a much greater importance than heretofore. Hitherto they had been observed, simply by the times, for the correction of the ephemeredes; now the chemical constituents of the sun can be determined, and the corona and red protuberances observed understandingly. Then to meet the new wants of astronomers, when our *Nautical Almanac* was enlarged in 1882 by Professor NEWCOMB, the data for eclipses were much improved; BESSEL'S tables and formulæ substituted for SAFFORD'S transformation; the path of totality given by the latitude and longitude of consecutive points; the duration of totality on the centre line given; the charts enlarged and improved, and quantities referred to the Greenwich meridian instead of that of Washington. This brings the subjects down to the present time.

2. *General Remarks on the Computation.*—This should be made for Greenwich mean time and the Greenwich meridian, for which all the quantities required are given, both in the English and American

* *Theorie de l'Eclipse du soleile et des phenomenes que s'y rattachent*, Leipsic, 1858.
† *Spherical and Practical Astronomy*, 1863.

Nautical Almanacs. If desired, all the results can be subsequently reduced to the local time and the meridian of any place. Although the present work is complete in itself, yet on account of the numerous references to CHAUVENET'S Chapter on Eclipses, the reader will find it to his advantage to have a copy of that work at hand,* in which also may be found the derivation of the several formulæ here quoted. For the numerical computation the formulæ and precepts in the present volume will be found all-sufficient.

It is recommended that the computer take up the parts of the work in the order here given by sections, on account of quantities in a preceding section being often required in those following. For the same reason the formulæ in one section are given in the order most convenient for use, followed by precepts for the computation.

The graphic method of representing quantities in the formulæ, which thus explains the whole theory of eclipses, has not, so far as the author is aware, been made use of in explaining an intricate series of equations, such as those for the principal times, the central line, outline, etc.

The computer should provide himself with good tables. BRUHN'S seven-place logarithms are the best, on account of the first six degrees being given to every second; without this it will be difficult to get the sines of the numerous small angles. ZECH'S *Addition and Subtraction Logarithms* (seven-place) are convenient, but as they are used only a few times, they may be dispensed with. GAUSS' five-place logarithms given to seconds, or NEWCOMB'S five-place logarithms given to decimals of a minute, may be used for the greater part of the computation; and when using four-place logarithms, they will be found much more convenient than the four-place tables, on account of there being no interpolations necessary. All operations are algebraic, and strict regard must be paid to the sign *before* a term, as well as to the *sign of the term itself*, which latter is dependent upon the signs of its factors. A beginner will find that many of his mistakes arise from disregard to these rules of algebra.

In such a computation as the eclipses, where signs change frequently, it is the author's custom to write the signs before each logarithm, whether the number be plus or minus; or before the first and last of each group. By so doing the omission of a sign is at once perceived. The sign may be omitted before a log. tangent, as the quadrant is determined by the signs before its sine and cosine.

* *Manual of Spherical and Practical Astronomy*, by WILLIAM CHAUVENET, Philadelphia, J. B. Lippincott & Co., 1863.

These computations, especially those for prediction, should be made with the greatest accuracy. They serve not only to warn the astronomer when to watch for his observed times, but also, when compared with the observation, serve to indicate the *accuracy of the tables*, which is of far greater importance.

No long computation can generally be made except upon paper ruled so as to keep the work in lines and columns. In the *Nautical Almanac* office several kinds are provided. That mostly used is in sheets of the best stout linen paper, 21 inches broad by 16 inches, ruled lengthwise into spaces about nine to the inch, and vertically eleven to two inches, which gives 85 lines to a page. One figure is usually placed in a space so formed. A solar eclipse will generally require three such sheets, and a lunar eclipse one side of a sheet.

Some of my readers may possibly charge me with having been too minute in the explanations and in giving the examples in full. The latter is the suggestion of a student in this work who wrote to me for explanations. An example given in full will generally show *how* each result is obtained; but *results only*, as examples are generally given, often fail to explain the difficulties a student meets with.

3. *Differentiated Quantities.*—In the matter of the graphic representation of a formula as made use of in this work, the author is of opinion that, except in very simple cases, formulæ derived from differentiation cannot always be shown graphically *in the same manner* as the primitive equation; especially after they have been much transformed or simplified. The reasons why this cannot be done are various. Differentiated quantities are usually so small that they cannot be shown upon the same scale as the terms which compose them. Another reason may be that the equation has various values, depending upon the value of differential of the independent variable, and this value may not at once be evident; that is, we may have its value in figures, but cannot use them in a strictly graphical manner. To illustrate this, take the following example from the eclipse formulæ:

$$x = r \cos \delta \sin (a - a).$$

Differentiate

$$dx = r \cos \delta \cos (a - a)\ d(a - a)$$
$$= 0.001018\, r \cos (a - a).$$

$d(a - a)$ in CHAUVENET's example is about 35 minutes of arc, which in parts of radius gives the decimal in the third equation. Now we can easily show x graphically from its factors. We can also show

the factors $r \cos \delta \cos (\alpha - a)$, either separately or combined; hence, dx must be about one-hundredth part of this latter. But this is not strictly a *graphical* representation, but *numerical*, and we do not show dx "*in the same manner*" as we do x. In this equation, r has a value of about 60 units; this and the small numerical term cannot possibly be shown on the same scale as above stated.

Where a differentiated equation has been transformed, the quantities become so estranged, as it were, that it is nearly impossible to follow them; nevertheless some expedient may generally be found to *illustrate* the required differential. In this manner we can show dx; it is simply the difference between two values of x computed for two successive hours, or the *hourly motion* of x; and it is shown in several of the subsequent figures when successive values of x are given.

4. *The Darkness at the Crucifixion.*—Before closing this section, the author may be pardoned for a few words upon the arguments of certain atheists and others who deny the miraculous darkness at the crucifixion, stating that it was caused by a total eclipse of the sun. But this could not possibly have been the cause of the darkness. The Jewish months were lunar, and commenced at new moon. Abib was the first month; preparations were made on the tenth day, and the Passover eaten on the fourteenth day (Ex. xii. 2, 6; Deut. xvi. 1). The Crucifixion took place at the time of the Passover, when the moon was *full*, now commonly called the *Paschal Full Moon;* and the darkness was not caused by an eclipse of the sun, which can take place only at *new* moon.

The author, having computed the eclipses yearly for the *Nautical Almanac* for twenty-three years, a longer period than that of any of his predecessors, resigned that portion of his work after completing the computation of the eclipses for 1905.

PART I.

SOLAR ECLIPSES.

SECTION II.

THE CRITERION.

5. AT the very threshold of this subject we meet with an article without which CHAUVENET'S chapter would be incomplete, and yet one which the practical computer will seldom use—an evidence of the completeness of his work.

The object of this section is merely to ascertain whether a solar eclipse will occur at or near a given time. Eclipses are not scattered irregularly through the year, but can occur only when the sun is near one of the moon's nodes. These have been styled by Professor NEWCOMB as *eclipse seasons*, the middle of which are the instants the sun passes the longitude of the moon's node; the seasons are therefore about six months apart, and a solar eclipse can occur only when the sun is within 18 days of the node, before or after; and a lunar eclipse 11½ days before or after.*

This precept is the most convenient we have for commencing our search, for which we will take the year 1902. Towards the close of Part I. of the *Nautical Almanac,* which is given for the Greenwich meridian, page 284 for this year, headed MOON, we find, "☊ Mean Longitude of Moon's Ascending Node." It will be noticed that it has a retrograde motion of about 20° in a year. The descending node ☋ differs 180° from the values here given. Now, comparing this with the Sun's Longitude, on page III., for each month, we find that they are the same on the following dates:

$$\begin{array}{llll} \text{Page 58, 1902, April 25,} & l' = ☋ = 214° - 180° = 34° \\ \text{" 166, \quad Oct. 19,} & l' = ☊ = \qquad\qquad 205 \end{array}$$

These dates are the middle of the eclipse seasons, and the seasons themselves are from April 7 to May 13, and Oct. 1 to Nov. 6.

We are now prepared to use CHAUVENET'S Criterion, from which we have, in addition to the general notation in this work, the following:

* *Popular Astronomy,* pages 29, 30.

l'	Sun's true longitude.
l β	Moon's true longitude and latitude.
$\Delta l'$ Δl $\Delta\beta$	Motions in 12 hours.
t	Interval of time to conjunction.

6. If, at the time of conjunction in longitude, we find, regarding the moon's latitude,—

$$\begin{array}{ll} \beta < 1^\circ\ 24'\ 34'', & \text{Eclipse is certain,*} \\ \beta > 1\ \ 34\ \ 47, & \text{Eclipse is impossible,} \\ \multicolumn{2}{l}{\text{Between these limits, Eclipse is doubtful.}} \end{array} \tag{1}$$

If the eclipse is very doubtful, by the value approaching the larger limit, we may then use the following more accurate formula:

$$\beta < \pi - \pi' + s + s' + 25'', \text{ Eclipse certain.} \tag{2}$$

We will pass over the ascending node and examine the other one. On pages 280–83 of the *Nautical Almanac* we have the moon's latitude and longitude, and on page 58 these quantities for the sun. We need only look near these dates when the moon's latitude is near 0. We therefore have the dates April 9 and May 6 to examine for solar eclipses and April 23 for a lunar. For this latter the sun's longitude will differ 180° from that of the moon, and we will neglect this until we take up Lunar Eclipses. From these pages we find for 1902 May 7 12^h.

$$\begin{array}{lll} \text{For the sun,} & l' = 46^\circ\ 28' & \\ \text{For the moon,} & l\ = 47\ \ 12 & \beta = -1^\circ\ 10'\ 32.7'' \end{array}$$

We see by inspection, without doing anything further, that there is an eclipse at this date. Passing to the other date, April 23, we take from the *Almanac* to the nearest minute as follows:

Date.	l'.		l.		$l-l'$.		β.	
April 8 . .	17° 44′		16° 41′		−1° 3′		+1° 36′	
		+29		+7 18		+6 49		−39
" 8.5 . .	18 13		23 59		+5 46		0 57	
		29		7 21		6 52		40
" 9 . .	18 42		31 20		12 38		+0 17	
		+30		+7 22		+6 52		−41
" 9.5 . .	19 12		38 42		+19 30		−0 24	

The sun's longitude is here interpolated to 12 hours, and the difference $l - l'$ gotten in the third group. Now when the sun and moon are in conjunction, the difference $l - l'$ must be zero. If t is the interval from the first date, April 8, to the time of conjunction, and $\Delta l'$ Δl the motion between successive dates, we must have

$$l' + t\ \Delta l' = l + t\ \Delta l.$$

* I have slightly changed CHAUVENET's values here for a reason given on a subsequent page of this section.

Whence

$$t = \frac{l' - l}{\Delta l - \Delta l'}, \tag{3}$$

which is the usual formula for interpolation, omitting second differences. In the present case we have

$$t = \frac{17^\circ\ 44' - 16'\ 41''}{7^\circ\ 18' - 29'} = \frac{1^\circ\ 3'}{6^\circ\ 49'} = +\ 0.15.$$

We may check this time as follows to ascertain if the longitudes are the same:

$$\begin{array}{ll} \text{For the sun,} & 17^\circ\ 44' + .15(29) \quad\;\; = 17^\circ\ 48', \\ \text{For the moon,} & 16 \quad 41 \;\; + .15(7^\circ\ 18) = 17 \quad 47', \end{array}$$

which is quite near enough for our purpose.

Now interpolating β for this interval, we have

$$\beta = +\ 1^\circ\ 36' - 6' = +\ 1^\circ\ 30'.$$

Comparing this with the Criterion equation (1), we find the latitude is between the limits, and rather doubtful; so we will now look to equation (2).

7. The above time 0.15 is a fraction of 12^h, and in hours is 1.80, which is the time of *New Moon*, given in the *Almanac* on page XII. for each month. It agrees very well with the time given there, $1^h\ 50^m.1$. For this time we must interpolate the several quantities. The sun's parallax is given on page 285, the semidiameter on page I., for each month; and the moon's parallax and semidiameter on page IV. for each month. The semidiameters given in the *Almanac* are marked *apparent* semidiameters, because they are affected by a constant of irradiation; this must be numerically deducted after interpolation. It has been changed sometimes, but its value may be found in the Appendix of the *Almanac*. Under the elements of the eclipses these quantities are called the *true* semidiameters because they are so. Interpolating from the above pages, we find

Moon's parallax,	π	$60'\ 1''.1$
Sun's parallax,	π'	$8\ .8$
	$\pi - \pi'$	$59\ 52\ .3$
Sun's true semidiameter,	$s'\ 15'\ 59''.2 - 1.15 =$	$15\ 58\ .1$
Moon's true semidiameter,	$s\ 16\ \ 22\ \ .5 - 1.50$	$16\ 21\ .0$
Constant,		$0\ 25\ .0$
Sum		$\beta = 1^\circ\ 32'\ 36''$

We seem to be worse off than we were before, for this value, which is more nearly accurate than the previous, being nearer to the outer limit, is more doubtful. Recourse must now be had to the rigorous method; but we will not pursue this example further. There is, in fact, a small partial eclipse at the north pole, and it may interest the reader to know how it has diminished during the past fifty years. The magnitudes are as follows:

1848,	March 6,	magnitude 0.269	
			− 53
1866,	" 16,	" 0.216	
			− 75
1884,	" 26,	" 0.141	
			− 76
1902,	April 8,	" 0.065	
			− 76
1920,	" 18,	" − 0.011	

Sun's diameter 1,000

The last value here in 1920 is assumed from the differences to accord with those above, and the negative sign of 0.011 shows that there will be no eclipse, this being the distance between the moon's shadow and the earth. The quantity is only approximate.

Should the reader desire to compare the time of conjunction computed above with that given in the *Almanac* for this eclipse, he must remember that we computed the conjunction in longitude, whereas the *Almanac* gives it in right ascension; these times are not the same, and the semidiameters and parallaxes would also differ; moreover, the above values are not rigorously interpolated.

8. *Rigorous Formulæ.*—No example of this will be here given, as it will probably be seldom required. The preliminary work is as follows:

Take from the *Nautical Almanac* the data as we did in the previous example, but retaining all the decimals. Copy the moon's longitude and latitude to include at least two dates beyond the times of conjunction and the time when the latitude is zero. Copy the sun's longitude for at least three dates each side of conjunction; that is, three days, interpolating carefully to 12 hours. Difference these and form the column $l - l'$ as before. Compute the time of conjunction as before, but using the formula given in the next section under the *Elements*, and used there for conjunction in right ascension. Second differences must be used here throughout, for which Tables II. and III. may be used, and at least one decimal of a second retained. This time of conjunction may be checked by interpolating the two longitudes, and they should agree *exactly*. Also interpolate the latitude exactly, using five-place logarithms, for the terms depending upon first differences.

We next have the following formulæ and notation:

$$\tan\ I = \frac{\tan \beta}{\sin(\Omega \sim l)} = \frac{\tan \Delta\beta}{\sin \Delta l} \tag{4}$$

$$\tan\ I' = \frac{\lambda}{\lambda - 1} \tan I \tag{5}$$

$$\beta \cos I' < \pi - \pi' + s + s' \tag{6}$$

I Inclination of the moon's orbit to the ecliptic.
I' An auxiliary angle.
λ The quotient of the moon's motion in longitude divided by that of the sun.

9. CHAUVENET does not show how to get the inclination I, nor is it given in the *Almanac;* but it is found as above by the simple right-angled triangle between the moon and the node, either before or after its passage. The longitude of the node is the same as the moon's longitude at the instant when the latitude is zero. It is given in the *Almanac* to the tenth of a minute, but as it may be required closer than that, it can be computed. It will be a little more convenient, instead of computing the node and then the term $\Omega \sim l$, to use the second form, in which Δl and $\Delta\beta$ are the differences for 12 hours, just at the node.

For λ the differences at the time of conjunction must be used, computing by logarithms to five places. With ZECH's Logarithms, the subtraction table with the argument log λ, gives at once log $\frac{\lambda}{\lambda - 1}$, and thence tan I'. Cos β is already computed, hence the first member $\beta \cos I'$.

Then from the *Almanac* take out the moon's semidiameter and parallax, page IV., for the month; the sun's semidiameter, page I., and the parallax at the close of Part I., page 285, or thereabouts; all of which must be interpolated carefully to the time of conjunction, using second differences where necessary. The semidiameters must next be corrected by deducting the constants of irradiation, which have been changed at times, but which are given in the appendix of the *Nautical Almanac.* Those that have heretofore been used are given in the next section, under the head of *Constants.* The second member of equation (6) may now be formed, and the comparison made; if the first member is the least numerically, an eclipse will occur.

I would hardly recommend this method, except on rare occasions.

Generally there are other methods of discovering an eclipse, but if they cannot be used, this method is lost work if there is an eclipse; so it might be better to suppose there to be one, and proceed by the regular formulæ given in a subsequent section, computing the eclipse tables for only three hours as far as l for penumbra. Then if the first approximation discloses an eclipse, compute the tables for other hours that may be required, and proceed in the regular way. There will then be no loss of the work done.

10. *The Semidiameters and Parallaxes.*—The values of these given by CHAUVENET, on page 438, seem to need revision, the moon's least parallax being in error by at least one minute. I found that authorities are not agreed as to the moon, and then consulted Professor RUEL KEITH, who has computed the moon's semidiameter and parallax for the *Nautical Almanac* for thirty consecutive years; and his reply was, that "the moon changes its distance so irregularly it is hard to follow it by rule." My only recourse, and I believe it is the best authority yet, is to take the extreme values of the moon's parallax from all the eclipses I have computed from 1883 to 1905, both years inclusive. This I have in a bound volume, giving the full data of all these eclipses. These extremes may not be the greatest possible, but they are at least better than those heretofore given as correct, for the greatest value is greater than that given by some authorities, and the least value is less. The authorities for the Moon's Parallax are as follows:

AUTHORITY.	Greatest Value.	Least Value.	Mean Value.
CHAUVENET, Astronomy, 1862; WOOLHOUSE (BARTLETT), 1836; PEIRCE, Trigonometry, 1852 . .	61′ 32.″	52′ 50.″	57′ 11.″
LARDNER, Handbook, Astronomy . .	61 18.	53 58.	57 38.
YOUNG, Astronomy	61 28.	53 55.	57 2.
PROCTOR, on the Moon	61 28.8	53 51.5	57 2.7
NEWCOMB, Astronomical Constants . .	. . .	. . .	57 2.55
Eclipse values 1883–1905	61 27.3	53 55.9	
HANSEN'S Tables 1880	61 27.8		

PEIRCE wrote in 1852: he was the principal mathematician connected with the *Nautical Almanac* when it was first published—about that time. WOOLHOUSE'S method was devised for the English *Nautical Almanac* in 1836, and is given in BARTLETT'S *Astronomy*. The eclipse values are the greatest and least values which have actually

occurred at the time of any eclipse during the years 1883 to 1905 inclusive. It seems as if WOOLHOUSE, the earliest writer, had decreased the least value and increased the mean so as to make it a *middle* value, which it properly is not.

The value 61′ 27″.8 occurs in 1880, Dec. 31, and very nearly in 1899, Jan. 11; and 53′ 55″.9 in 1901, Nov. 11, and also very nearly this value in 1883, Oct. 30. I have not had occasion to examine other portions of these years, and cannot say that these are the greatest and least that have occurred, but they are so during the eclipse seasons. The parallaxes repeat themselves with the Saros, from which I found the value 61′ 27″.8. Therefore, assuming values a little beyond the greatest and least, and with NEWCOMB's mean, we take the following as the standard values for the moon:

Greatest, 61′ 28″.8. Least, 53′ 55″.0. Mean, 57′ 2″.52.

And the semidiameters are gotten by the formula:

$$s = k\pi = 0.272274\ \pi.$$

For the sun, taking from NEWCOMB's Tables of the Sun * the semidiameter, which is AUWER's value, and parallax, we have the following:

ADOPTED VALUES.

	Greatest Value.	Least Value.	Mean Value.
Moon's equa. hor. parallax π	61′ 28″.8	53′ 55″.0	57′ 2″.25 †
Sun's equa. hor. parallax π'	8 .9	8 .6	8 .80
Moon's true semidiameter s	16 44 .4	14 40 .8	15 31 .79
Sun's true semidiameter s'	16 16 .0	15 43 .8	15 59 .63

And from these we obtain the values given above in formula (1).

$$\beta = 1^\circ\ 24'\ 10''.7 + 23''.8 = 1^\circ\ 24'\ 34''.5$$
$$\beta = 1\ \ 34\ \ 20\ .6 + 26\ .7 = 1\ \ 34\ \ 47\ .3$$

* *Astronomical Papers*, prepared for the use of the *American Ephemeris and Nautical Almanac*, vol. vii., part i. The above constant of the sine of lunar parallax expressed in arc, 57′ 2″.52, is also used in these tables, p. 12.

† HANSEN's parallax is used in the *Nautical Almanac*. This value of sin parallax is given by Professor NEWCOMB in his *Transformation of Hansen's Lunar Theory*, *Astr. Papers*, vol. i., p. 105, sin π 57′ 2″.09. The arc is 0″.16 greater, as given in the text. In his *Astronomical Constants*, p. 194, Professor NEWCOMB has deduced the value 57′ 2″.52.

The mean of the small terms is almost exactly 25″, as CHAUVENET gives it.

11. *The Saros.*—This term is derived from the ancient astronomers, but is now applied to the period of the recurrence of eclipses. It is given in NEWCOMB'S *Popular Astronomy* thus:

242 returns of the moon to her node . . . . 6585.357 days
19 returns of the sun to moon's node 6585.780

The sun and moon are therefore together 223 times during this period, the length of which is, according to Professor NEWCOMB in his *Recurrence of Solar Eclipses,**

$$6585.321222 \text{ days},$$

or

$$18^y \, {}^{10}_{11}{}^d \, 7^h \, 42^m \, 33^s.6, \text{ if Feb. 29 intervenes} \begin{Bmatrix} 5 \text{ times}, \\ 4 \text{ times}. \end{Bmatrix} \qquad (7)$$

In searching for an eclipse for the years 1882 to 1899 we must use for the Saros the following:

$$18^y \, {}^{10}_{11}{}^d \, 12^h \, 51^m \Big\} \qquad (7 \textit{ bis.})$$

because the eclipses before 1882 are given to Washington mean time, and this value gives the reduction to Greenwich mean time from 1882 to 1899 inclusive. After the latter date the value in equation (7) should be used.

Also, as stated above, there being 19 conjunctions of the sun with the same node, if we divide the above number by 19, we get the interval between them, 346.59 days. Hence, the sun will return to the node,

$$365.25 - 346.59 = 18.66 \text{ days},$$

earlier each year.† These are mean values, and may vary from the actual times, but perhaps not over two hours.

It is a remarkable circumstance connected with the Saros that the longitude of the moon's perigee is nearly the same after 18 years. The sun also is within 11° of its former place, consequently all the

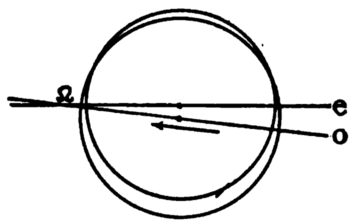
* *Astronomical Papers*, vol. i., part i., p. 17.

† *Newcomb's Popular Astronomy*, 1878, p. 30, has a misprint giving this wrongly as 19⅝ days, which puzzled me when I first took up the study of eclipses.

quantities affecting an eclipse—the parallaxes, semidiameters, declinations, hourly motions, etc.—will be very nearly the same as they were $18^y\ 11^d$ previously; and the eclipse also will therefore be very similar.

12. If the sun and moon start out together at a node, at the end of the Saros they will be together a little *before* they reach the node. And if this is the ascending node, as in Fig. 1, the moon in the orbit *o* will consequently be a little *below*, or *south* of the sun in the ecliptic *e*, and the reverse must be the case at the descending node; hence, we have a valuable and easy criterion:

FIG. 1.



$$\left.\begin{array}{l}\text{At Moon's Ascending Node, series is moving South,}\\ \text{" Descending " " " North.}\end{array}\right\} \quad (8)$$

This holds good for both solar and lunar eclipses.

The moon's nodes can be found from the *Nautical Almanac*, pages 280–83, or thereabouts. When the latitude is zero, the differences are *positive* at the ascending node and *negative* at the descending node.

13. We will take the year 1904 to illustrate the period of the Saros. By reference to the *Nautical Almanac* of eighteen years previously, we find under the "Eclipses in 1886" that there are but two, both solar; selecting the August eclipse we find

Greenwich mean time of ☌ in R. A.,	1886, Aug.	29^d	0^h	59^m
And the Saros as above (Feb. 29 4 times),	18^y	11	7	42
	1904, Sept.	9	8	41

which is near enough to the true time for us to take out our data from the *Almanac*. In fact, as this is an example, we may say that it is correct within 10 minutes of the correct time, which is closer than it generally comes.

14. In making search for the eclipses for a given year the best method will be: *First*, to find all the eclipses, solar and lunar, as given by the Saros, as above. *Second*, to ascertain whether any have run out. The example given above and the comparison of magni-

tudes show that no partial solar eclipse whose magnitude is as much as 0.100 will be likely to run out. Sometimes the shadow moves more slowly. In case of very small magnitudes, refer back 36, or preferably also 54, years, and very likely the case can be settled; or else employ the formulæ (1) or (2). *Third*, to find whether any new series enters. This is more difficult, and the computer must be absolutely certain about this point. If he possesses access to OPPOLZER'S *Canon of Eclipses* (in German), he can find out what eclipses will occur in the future, examining especially the year he is about to compute for. It would be well also to look 18 years ahead, and if a new series is found to enter, then it would not be safe to assume that it will *not* appear during the year he is searching for, because OPPOLZER'S method of computing is approximate, and *may* not have caught the very first appearance of a new series. If the matter is still doubtful, or the computer does not possess a copy of OPPOLZER, he should know where to look for a supposed eclipse. Only three eclipses can possibly occur at each node, two solar and one lunar, or the reverse: one large eclipse very near the node and two partial eclipses, about fourteen days, one on each side of the node. So if there is a large eclipse at the node and *one* partial, fourteen days distant, there *may* be another partial on the other side. It sometimes occurs that there is but one eclipse at each node, as in the year 1904. In this case a lunar may enter at either node. This occurs, according to OPPOLZER, in 1958, May 3. If there are two eclipses, more or less seven days from the node, there can be no other. Formula (1) or (2) should be used in these cases.

There is one other point to be noted: if the eclipse season occurs in December, an eclipse may occur in the January following, and, *vice versa*, the middle of the season occurring in January of the next year *may* cause an eclipse in December of the present year.

And generally under very doubtful cases, as I suggested above, I would proceed as if there were an eclipse rather than go through the rigorous method for criterion. In this case, if there is no eclipse, it will show itself by sin ψ in the formulæ for beginning and ending coming out greater than unity, which is impossible, and the computation can be carried no farther.

SECTION III.

DATA AND ELEMENTS.

15. THE data for a solar eclipse are required for six, seven, or eight hours of Greenwich mean time, according to the size of the eclipse, and including the time of conjunction in right ascension. We will proceed as follows: Taking the time of conjunction as given by the Saros in the previous section, compare the sun and moon's right ascensions, interpolating the sun's by the hourly motions to seconds only, and it will soon be found between what hours the conjunction lies. Then with these values of the sun and moon find the time within a few minutes by formula (3) used above, or by that used for the elements in this section. This gives us pretty nearly the time of conjunction of the present eclipse, and we can now find out how many hours we will have to compute for in the following manner:

In the previous section we found the time of an eclipse to be by the Saros 1904, September $9^d\ 8^h\ 41^m$, and by comparing the right ascensions as above, we find a more correct time to be, we will suppose, $8^h\ 47^m$. Now take the previous eclipse as follows:

		1886, August.		1904, September.	
Eclipse begins 1886, Aug.		$28^d\ 22^h\ 18^m$		$9^d\ \ 6^h\ \ 1^m$	
			2 40		2 40
☌ in R. A.,	"	29 0 58		8 41	
			2 34		2 34
Eclipse ends	"	29 3 32		11 15	

Apply the intervals of the eclipse of 1886 to the time of conjunction of 1904, and we have very nearly the times of beginning and ending, so that we will have to compute for seven hours—6 to 12, both inclusive.

Therefore, copy from the *Nautical Almanac* as follows:

Sun's R. A. and Dec.	six dates	from p.	II.	for the month,	
log radius vector	"	"	III.	"	"
semidiameter	"	"	I.	"	"
Sidereal time	one date	"	II.	"	"
Moon's R. A. and Dec.	for the eclipse hours,		V.–XII.		"
semidiameter,	six dates of 12 hours,		IV.	"	"

16. These six quantities are required for all eclipses for interpolation. The sun's parallax can be taken from page 285 for the previous year, if this year is not in print. It is sensibly constant for the same day of each year. This and the semidiameters are used only as checks.

The quantities should be carefully differenced, and then interpolated by the differences for each of the eclipse hours (the semidiameters excepted). They should be interpolated by the usual formula as follows, applied to each of the eclipse hours :

$$a_t = a_0 + t\Delta_1 + \frac{t(t-1)}{1 \cdot 2}\Delta_2. \tag{9}$$

The sun's right ascension should be carried out to three decimals of time and the declination to two decimals of arc, also the moon's parallax to three decimals of arc ; log r depending upon it will then difference smoothly, which is a great advantage ; otherwise it will not.

The interpolations are not difficult, but require care. The several values of the second term of the formulæ when applied to each hour are multiples of $\frac{1}{24}$ of the first differences of the quantities copied, as above, and the third term can be taken from Table II. of this work. This table, which the author has used for a number of years, is in two parts—for right ascension in time and for declination in arc; the columns give multiples of the coefficient depending upon the argument Δ_2. Log r can be taken out of either table, considering the quantities as whole numbers instead of as decimals in the margin, and moving the decimal points in the body of the table. For the moon's parallax, which is given to 12 hours, the table can also be used, considering that 2, 4, 6 hours in the table represent 1, 2, 3, since $\frac{2}{24} = \frac{1}{12}$, and so on. This term, as is well known, is always negative, and, supposing the interpolations performed, we have the data here given with the differences.

17. The following is all the data required, except constants, to compute a solar eclipse. It is exactly as I have it in my computing sheets for the *Nautical Almanac*, and the reader will notice that I do not repeat figures unnecessarily. The reduction from time to arc can be performed by means of Table I.

Data for Total Solar Eclipse, 1904, *September* 9.

Gr. M. Time.	☉ Apparent R. A.	Diff.	☽ Apparent R. A.	1st Diff.	2nd Diff.	log r′.	Diff.	π.	1st Diff.	2nd Diff.
6^d	11^h 10^m $39^s.955$	+ 9.001	11^h 4^m $11^s.95$	+ 2 26.38	— .10	0.0029469	— 47	61′ 22″.981	+ .025	— .036
7	48 .956	.000	6 38 .33	26.28	9	9422	47	23 .006	— .011	35
8	57 .956	.000	9 4 .61	26.19	9	9375	47	22 .995	46	34
9	11 6 .956	.000	11 30 .80	26.10	9	9328	48	22 .949	80	35
10	15 .956	8.999	13 56 .90	26.01	— .10	9280	47	22 .869	.115	— .039
12	24 .955	+ 8.999	16 22 .91	+ 2 25.91		9233	— 47	22 .754	+ .154	
11	11 11 33 .954		11 18 48 .82			0.0029186		61 22 .600		

	α′.	Diff.	α.	1st Diff.	2nd Diff.	α — α′.	1st Diff.	2nd Diff.	3rd Diff.
6	167° 39′ 59″.33	+ 2 15.01	166° 2′ 59″.25	+ 36 25.70	— 1.50	— 1° 37′ 0″.08	+ 34 20.69	— 1.49	+ .14
7	42 14 .34	.00	39 34 .95	36 34.20	1.35	1 2 39 .39	34 19.20	1.35	0
8	44 29 .34	.00	7 16 9 .15	36 32.85	1.35	— 0 28 20 .19	34 17.85	1.35	+ 3
9	46 44 .34	.00	52 42 .00	36 31.50	1.35	+ 0 5 57 .66	34 16.50	1.32	— .21
10	48 59 .34	14.99	8 29 13 .50	36 30.15	— 1.50	0 40 14 .16	34 15.18	— 1.53	
11	51 14 .33	+ 2 14.98	9 5 43 .65	+ 36 28.65		1 14 29 .34	+ 34 13.65		
12	167 53 29 .31		169 42 12 .30			+ 1 48 42 .99			

	δ′.	1st Diff.	2nd Diff.	δ.	1st Diff.	2nd Diff.	δ — δ′.	1st Diff.	2nd Diff.	3rd Diff.
6	+ 5° 17′ 36″.60	+ 56.72	+ 0	+ 5° 36′ 58″.7	— 11 27.3	— 2.2	+ 0° 19′ 22″.10	— 10 30.58	— 2.20	+ .12
7	16 39 .88	56.72	2	25 31 .4	11 29.5	2.1	+ 8 51 .52	10 32.78	2.08	.08
8	15 43 .16	56.74	0	14 1 .9	11 31.6	2.0	— 1 41 .26	10 34.86	2.00	.11
9	14 46 .42	56.74	1	2 30 .3	11 33.6	1.9	12 16 .12	10 36.86	1.89	+ .01
10	13 49 .68	56.75	+ 2	4 50 56 .7	11 35.5	— 1.9	22 52 .98	10 38.75	— 1.88	
11	12 52 .93	+ 56.77		39 21 .2	— 11 37.4		33 31 .73	— 10 40.63		
12	+ 5 11 56 .16			+ 4 27 43 .8			+ 0 44 12 .36			

Sidereal time, Sept. 9 at noon, 11^h 12^m $24^s.98$ = 168° 6′ 14″.70.

18. There is a principle in the theory of differences that I have never seen in any of the ordinary works on interpolation, and that is, the symbol Δ is *distributive.* We have for multiplication in algebra $m(a \pm b) = ma \pm mb$, where m is distributive. Likewise in differences,

$$\Delta(a \pm b) = \Delta a \pm \Delta b. \tag{10}$$

Now, if a and b are similar terms of two series and the series be added term by term, the corresponding difference of the sum will be the sum of those differences. This is illustrated by the first differences of the two right ascensions given above; compare their sum term by term with the differences of $a - a'$. Likewise compare the second and the third differences. We can check a number of additions in this manner. This principle I have found of very great use in certain methods of computation for shortening the work.

19. There is also another point similar to the above, the moon's right ascension differences very irregularly when reduced to arc; this depends upon the fact that in differentiating, a constant factor in the primitive remains in all the differences. Thus,

$$u = ax^n \qquad \frac{du}{dx} = anx^{n-1} \qquad \frac{du^2}{dx^2} = an(n-1)x^{n-2}.$$

Now reduction to arc is simply a multiplication by 15, and the differences of a are seen to be exactly 15 times those of ☽ App. R. A. The moon's declination, given to tenths of a second, causes $\delta - \delta'$ to vary irregularly by multiples of 10, which are slightly affected by the sun's differences. Both of these points will be found of much assistance on a subsequent page.

20. *Eclipse Constants.*—The principal of these are π_0, the sun's parallax, H, the sun's mean semidiameter, and k, the ratio of the moon's equatorial radius to that of the earth. They have been changed from time to time as more accurate values have been obtained.

As to the formulæ and constants used in past years, I am told by Mr. Hill that Mr. Chauncey Wright, who first computed the *Almanac*, used his own formulæ, as did Mr. Hill, who succeeded him for the year 1874.

Chauvenet, for his eclipses, has, through an oversight, used *both values* of Encke's Parallax of the Sun. For the Constants of the Cones, on page 448, he has used $\pi_0 = 8''.57116$, which is Encke's *second* value, and which is satisfactory; but on page 452 he gives log

sin $\pi_0 = 5.61894$, which belongs to ENCKE's *first* value, 8.5776. This latter is used for finding b, and also enters into the constants for the cones; but the error is only about eight (8) units in seven-place logarithms, and hardly affects the radii of shadows.

Since the author has computed the eclipses, the following constants have been used for the eclipses of the years given:

1883, CHAUVENET's values as follows:

π_0, ENCKE's 2d, for constants for cones,		8''.57116	log sin	5.61861
1st, for b,		8 .5776	log sin	5.61894
H, BESSEL's value,		959 .788	log	2.9821753
k, BURCKHARDT's value,		0 .27227	log	9.4349998
Resulting constants for cones are		Penumbra	log C	7.6688033
		Umbra	log C_1	7.6666913
1896, π_0,		8.80	log	5.63006
Constants for cones,	log C	7.6688314	log C_1	7.6666631
1901, when NEWCOMB's new tables were generally first used,				
H,		959.63	log	2.9821038
k,		0.272274	log	9.4350062
Resulting constants,	log C	7.6687600	log C_1	7.6665914
For 1902–3–4 only,				
k,		0.272506	log	9.4353760
Resulting constants,	log C	7.6687609	log C_1	7.6665909
For 1905 to the present time,				
k, the previous value restored,		0.272274	log	9.4350062
Resulting constants,	log C	7.6687600	log C_1	7.6665914

Constants of Irradiation.

1883–1900,	Sun, 2''.20	Moon, 2''.50
1901,	1 .15	2 .50
1902,	1 .15	1 .50
1903–04,	$\frac{1\ .87}{r'}$, variable.	1 .50
1905,	$\frac{1''.87}{r'}$, variable.	1 .50 + 0.000232 π

The two latter are variable; their values may be taken from the small tables given in the next article. These quantities are not, strictly speaking, constants, for they vary with the state of the atmosphere, the telescope, and the eye of the observer, and the above values are arbitrarily assumed.

The following constants are also used in eclipse computations:
Eccentricity of the terrestrial spheroid e,

CHAUVENET (BESSEL'S) value, 0.0816967		log 8.9122052
For 1882 and following years CLARKE,		8.9152515
	$\log \sqrt{1-e^2}$	9.99853
	$\log \frac{1}{\sqrt{1-e^2}}$	0.00147

These constants should be written on the lower edge of a slip of paper for future use, as they are not repeated where used in the examples; and placing the constant above any quantity in the examples, it can be added or subtracted without rewriting it.

In 1892, for the eclipses of 1896, Professor NEWCOMB substituted the parallax 8.80 for ENCKE'S, which was afterward adopted by the *Paris Conference* in 1896; and it was first used in the body of the *Almanac* for 1900. The changes in H have been small, and for k substantially the same value until 1899, for the eclipses of 1902, after Professor NEWCOMB'S retirement, when the value was unfortunately much increased to 0.272506 (log 9.4352760), a value derived chiefly from *occultations*. Compare CHAUVENET, I., 551, *note:* "According to OUDEMANS (*Astron. Nach.*, vol. li., p. 30), we should use for occultations $k = 0.27264$, or $\log k = 9.435590$, which amounts to taking the moon's apparent semidiameter about $1''.25$ greater in occultations than in solar eclipses." Soon after the eclipse of 1900, May 28, a letter from the English *Nautical Almanac* office was received in our office, stating that our values of the Besselian elements gave the times almost exactly; whereupon the director promptly restored the old value of $k = 0.272274$. This comparison with our *Almanac* caused the English *Nautical Almanac* to adopt a smaller value of k, very nearly equal to our value.

Wishing to know for the present work the exact values the English *Nautical Almanac* would adopt, Mr. A. M. W. DOWNING, superintendent of the English *Nautical Almanac* office, in reply to my queries, very kindly wrote as follows, under date of 15 November, 1901, and in advance of the publication of the new values in the *Almanac* for 1905:

"In reply to your letter of 5th instant, the constant of lunar parallax used in the eclipse Computations is HANSEN'S, viz., $57'\ 2''.28$. The value of $\log k$ is given on page 629 (*N. A.*, 1904), and is 9.43542, corresponding to mean semidiameter of ☾ 15′ 32.65.

"But this semidiameter appears from the observations of the late total eclipse of the sun to be too large for use in eclipse calculations,

and in 1905 *Nautical Almanac* we adopt 15′ 31″.47, deduced by Dr. J. PETERS, of Berlin, from a discussion of recent observations of eclipses of the sun. We still use 15′ 32.65 for occultations of stars by the moon."

The above values differ slightly from ours, but the two *Almanacs* are generally now in accord upon this point. The value of k, giving the above semidiameters for eclipses from the above parallax, is $k = 0.272178$, $\log k = 9.4348533$.

21. *Elements.*—These are fundamental values for an eclipse, because from them alone the times can be computed for any place by the method of semidiameters; or the eclipse plotted graphically and the times measured by scale. These elements are the time of conjunction in right ascension, and for this time—the right ascensions, declinations, hourly motions, semidiameters, and parallaxes.

For the time, which is when $(\alpha - \alpha') = 0$, we have

$$t = \frac{-(\alpha - \alpha')}{\Delta_1 - \tfrac{1}{2}\Delta_2 - \tfrac{1}{2}\Delta_2 \dfrac{(\alpha - \alpha')}{\Delta_1}} \tag{11}$$

For this use all the decimals and six or seven places of logarithms. Δ_2 is here a mean of the two values adjacent to Δ_1, and the term $\dfrac{\alpha - \alpha'}{\Delta_1}$ is an approximate value of t, which can be gotten mentally. $(\alpha - \alpha')$ is the value just *before* conjunction. From the Data, page 35, t is roughly $\frac{28}{34} = 0.8$.

$+\Delta_1$	$+34'\ 17''.85$	$-(\alpha - \alpha')$	$+28'\ 20''.19$	log 3.2304974
$-\tfrac{1}{2}\Delta_2$ $\tfrac{1}{2}(1.35)$	$+\ 0\ .675$			
-0.8×0.675	$-\ \ .540$			
	$34'\ 17''.985$			log 3.3134423
$0.826143^h = 49^m.5686$				log t 9.9170551
∴ ☌		$8^h\ 49^m\ 34''.12$		

The result gives decimals of an hour, since the data are given by hours.

For this time the right ascensions and declinations are to be interpolated (Formula 9), using five-place logarithms and second differences for the moon, which can be taken from Table III. For the above decimal of an hour we find the coefficient for $\Delta_2 - 0.072$. The correction for Δ_2 can be gotten mentally. The right ascensions must agree *exactly*, which will check the time, at least the logarithm. The declinations can be checked, if desired, by interpolating $\delta - \delta'$, which should equal their difference very closely.

Next, interpolate $\log r'$ for the time t; it can generally be done mentally. Then the true semidiameter and parallax are

$$s' = \frac{H}{r'} \tag{12}$$

$$\pi' = \frac{\pi_0}{r'} \tag{13}$$

These can be checked by the values from the *Nautical Almanac*, interpolating for the time t, and deducting from the semidiameter the constant of irradiation mentioned above. This variable value for 1903 to the present time can be taken from annexed table.

Constant of Irradiation for Sun's Semidiameter.

$\log r'$.		Irradiation.
0.0094		1″.83
	−24	
70		1 .84
	24	
46		1 .85
	23	
28		1 .86
	23	
00		1 .87
	24	
9.9976		1 .88
	23	
53		1 .89
	23	
30		1 .90
	−22	
9.9908		1 .91

Next, interpolate the moon's parallax to three decimals for the time t by natural numbers, and get the logarithms in seconds; then the true semidiameter,

$$s = k\pi. \tag{14}$$

Check this by interpolating the moon's semidiameter from the *Nautical Almanac*, using second differences for an interval of 12 hours (Table III.), and deducting the constant of irradiation from the annexed table, or according to the value of the constant used. This should agree within about ″.05, because the *Almanac* gives only to tenths of a second, which may be ″.05 in error. The value here computed will be more correct than that in the *Almanac*, if no mistake is made.

Constant of Irradiation for Moon's Semidiameter.

π.		Constant Irradiation.	
52′ 50″	—	0″.73	— 1″.50
53 0		0 .74	1 .50
54 0		0 .75	1 .50
55 0		0 .77	1 .50
56 0		0 .78	1 .50
57 0		0 .79	1 .50
58 0		0 .81	1 .50
59 0		0 .82	1 .50
60 0		0 .83	1 .50
61 0		0 .85	1 .50
62 0	—	0 .86	— 1 .50

The hourly motions are gotten as follows: A formula can be used, but is unnecessary. For the moon in declination, we want the first difference at the time t, which is between 8 and 9^h. In the table of data $-11'$ $31''.6$ is the motion at 8.30, and $-11'$ $33''.6$ the motion at 9.30. ☌ is at 8^h 50^m nearly; that is, 20^m after the half hour; the change during the interval between the half hour is $\Delta_2 - 2''.0$; in 60^m, t is $\frac{20}{60}$ of this interval; and $\frac{1}{3}$ of -20, or -0.7, being added algebraically to $-11'$ $31''.6$, we have $-11'$ $32''.3$ for the hourly motion required.

Hence we have the elements—

Greenwich mean time of ☌ in R. A., 1904, Sept. $9^d\ 8^h\ 49^m\ 34^s.1$

Sun's R. A.	$11^h\ 11^m\ 5^s.39$	Hourly motion,	9.00
Moon's R. A.	11 11 5 .39	" "	2 26″.16
Sun's dec.	+ 5° 14′ 56″.29	" "	— 56″.7
Moon's dec.	+ 5 4 30 .69	" "	— 11′ 32″.3
Sun's equa. hor. parallax,	8′.74	True semidiameter	15′ 53″.17
Moon's " "	61 22′.957	" "	16 43 .63

I may add here that the time of new moon is not the same as that we have computed above, being the time of conjunction with the sun in *longitude*.

SECTION IV.

ECLIPSE TABLES.

22. With this section begins the main computation for the Eclipse Tables, with which all the rest of the calculation is made. Seven-place logarithms are to be used until l and log i are gotten; then five-place for the rest of the work.

The computer should prepare in advance the general constants for all eclipses on the lower edge of a slip of paper, so that it may be placed above other quantities it is to be combined with; and on another slip certain constants which change with each eclipse. Constants are generally not given in the examples following. The reader, however, following this suggestion, by the aid of the formulæ will doubtless have no difficulty with the examples, which are here given just as they stand on the author's computing sheets.

23. We recapitulate the general constants used in the examples as follows:

log π_0,	8.80		0.9445
log sin π_0,	8.80		5.63006
log H,	959.63		2.9821038
log k,*	0.272506		9.4353760
Constants for cones		Penumbra,	7.6687609
		Umbra,	7.6665909
log $\sqrt{1-e^2}$,			9.99853
log $\dfrac{1}{\sqrt{1-e^2}}$,			0.00147

* As stated above, this value was used only for the years 1902–3–4. The old and more correct value 0.272274 has now been restored.

A few other constants used only in particular places will be given in the text where they occur, and they can be added to the slip of paper on which the above should be written, as they are also used for all eclipses.

24. I propose to give first the formulæ in groups of related quantities, commenting upon them and their use in connection with the example which follows. In this order the computation will be given in an unbroken order. And following this will be given the Eclipse Tables which result; these are also necessarily explained with the formulæ. The small terms of a and d, and the quantities used only for them, may be computed with five-place logarithms.

25. *The Main Computation and* CHAUVENET'S *Formulæ.*—In this formula, G is the distance between the centres of the sun and moon,

$$G = r' - r.$$

And two auxiliaries are assumed, such that

$$\frac{G}{r'} = g \quad \text{and} \quad \frac{r}{r'} = b.$$

If we deduct the third of these equations from unity in each number, we have

$$\frac{r' - r}{r'} = 1 - b.$$

And since we have generally $\sin \pi' = \frac{\sin \pi_0}{r'}$, which is a more correct formula than No. (13), we have

$$b = \frac{r}{r'} = \frac{\sin \pi'}{\sin \pi} = \frac{\sin \pi_0}{r' \sin \pi}.$$

For the eclipse hours of Greenwich mean time, which were determined above, commence the computation with the following formulæ:

$$b = \frac{\sin \pi_0}{r' \sin \pi} \tag{15}$$

$$g = 1 - b \tag{16}$$

$$a = a' - \frac{b}{1 - b} \cos \delta \sec \delta' (a - a') \tag{17}$$

$$d = \delta' - \frac{b}{1 - b} (\delta - \delta') \tag{18}$$

a and d are the right ascension and declination of the point Z of the celestial sphere.

The quantity $1-b$ is not required separately, and by Table V. the quantity $\frac{1}{1-b}$ may be taken out at once with the argument $\frac{1}{b}$, or the compliment of $\log b$.

26. The second terms of formulæ (17) and (18) are very small—not over 10″ or 15″—and the numbers found from the logarithms are transferred to the Eclipse Tables and the sign changed. Then a and b are gotten there and differenced. As soon as transferred to the Eclipse Tables all quantities should be differenced, and the last difference should run at least as smoothly as in the present example.

$(a-a)$ and $(\delta-d)$ are next found and differenced, then $\frac{1}{2}(a-a)$ and $(\delta+d)$. The reader will remark that the difference of the small terms is equal to the third small term; thus,

$$(a-a')-(a-a')=a-a;$$

and similarly for $\delta-d$. This comparison will check the intermediate work.

27. The Greenwich hour angle of the principal meridian passing through the point Z at any time is given by the equation

$$\mu_1=\theta+h'-a. \tag{19}$$

This is done on the computing sheets. h' is the sidereal equivalent of the mean time eclipse hours h. Chauvenet adds $\theta+h'$ in time and then reduces to arc. The better way is to reduce the sidereal time to arc, which was done on the table of data already given; then use Table VI., which gives the sidereal equivalent in arc. The addition is made on the computing sheets, then μ_1 is given in the next column.

The hourly variation of μ_1 or the change in 1 hour in seconds of arc and in parts of radius,—

$$\log \mu_1'=\log(\Delta\mu_1 \sin 1'') \tag{20}$$

can be computed or taken at once from Table VII. with the argument $\Delta\mu_1$. It is constant for one eclipse, and is needed only to five decimals.

For the *Nautical Almanac* change of μ_1 for 1 *minute* of time in *minutes* of arc is

$$\log \Delta\mu \text{ now called } \log \mu'=\log \frac{\Delta\mu_1}{3600}. \tag{21}$$

$\Delta\mu_1$ is here given in seconds, and this is the change of μ_1 in minutes of arc for one minute of time. Given to four decimals only, and it can be taken from Table VII. at the bottom.

$$\log f = \log (\mu_1' \cos d) \tag{22}$$

Required to four decimals only.

This latter is from CHAUVENET'S formula 512, the rest of which is not needed at all,

$$f \sin F = d'$$
$$f \cos F = \mu_1' \cos d$$

in which placing cos $F =$ unity, we have the above form.

28. Proceeding with the computation,

$$r = \frac{1}{\sin \pi} \tag{23}$$

$$x = r \cos \delta \sin (a - a) \tag{24}$$

$$y = r \sin (\delta - d) \cos^2 \tfrac{1}{2} (a - a) + r \sin (\delta + d) \sin^2 \tfrac{1}{2} (a - a) \tag{25}$$

$$z = r \cos (\delta - d) \cos^2 \tfrac{1}{2} (\delta - a) - r \cos (\delta + d) \sin^2 \tfrac{1}{2} (\delta - a) \tag{26}$$

These are the coördinates of the axis of the cone of shadow, and they will be found rather difficult to get correctly. A partial check upon them is given under the article At Noon, but it is not of much value here, and given there chiefly as it leads up to other matters.

The second terms are small and computed with five-place logarithms; but added to the first terms by ZECH'S *Addition and Subtraction Logarithms.*

In the example following, and wherever else such tables are used, the expression $l - l$ means the difference of the two logarithms, usually expressed in the tables as "log — log." In addition this is called A, and the table gives B. In subtraction it is called B, and the table gives A.

Log r is given here, but is not on my computing sheets, but on a second slip of paper containing these quantities, which are constant for one eclipse, but change slowly with the season.

29. Copy log x in the Eclipse Tables and difference it, pass to numbers to six decimals and difference them also. Proceed the same with log y—but log z is not needed among the tables, so that it is not copied. It is required below in this computation, and it should difference smoothly if π is interpolated to three decimals.

x is seen to difference rather irregularly in the third differences, tracing this back, we find it in $(a - a)$ and still further back in a in arc; which, we said in the previous section, varies by multiples of 15, caused by the reduction from time to arc. To ascertain whether

the irregularities of x are due wholly to this cause or to small errors from other sources I have devised the following differential formula from No. 24. Differentiating, with x and a variable, we have

$$dx = -r \cos \delta \cos (a - \alpha)\, da \times \sin 1'' \qquad (27)$$

Assuming $da = 0''.01$, this table has been constructed from the various values of r and d. And to use it the reader is referred to the remarks following the Data of this eclipse given in Art. 18, Eq. 10. For this eclipse $\log r = 1.748$, $\delta = 5°$, whence from the annexed table we have 2.69; $\Delta_3(a - \alpha)$ in the tables varies thus, $-13 \; -2 \; +1 \; +14$. If we add to it the same series with contrary signs, $+13 \; +2 \; -1 \; -14$, it will all reduce to 0. Now every $0''.01$ affects x by 2.69 units in the sixth place of decimals. Multiply this latter series by this constant and add it to $\Delta_3 x$; the sum should reduce to zero, or more likely become some constant; we thus have

Differential Formula for x.

Log r	0°	10°	20°	30°
1.75	2.73	2.68	2.56	2.36
1.76	2.79	2.75	2.62	2.42
1.77	2.86	2.81	2.68	2.47
1.78	2.92	2.88	2.74	2.53
1.79	2.99	2.94	2.80	2.59
1.80	3.06	3.01	2.87	2.69
1.81	3.13	3.08	2.93	2.71

$\Delta_3 x$	-24	-54	-56	-96
$-2.69\,\Delta_3(a-\alpha)$	-35	-5	$+3$	$+38$
Sum	-59	-59	-53	-58, a constant, nearly,

which shows that the irregularity of x depends wholly upon omission of a third decimal in a in time.

A formula for y is similarly derived and here tabulated, and a formula for z may also be derived; but this latter is of little use, since z differences quite smoothly, or should do so if π is interpolated to three decimals.

Differential Formula for y.

Log r'.	Equation.	
1.750	2.73	
1.755	2.76	3
1.760	2.79	3
1.765	2.82	3
1.770	2.85	3
1.775	2.88	3
1.780	2.92	4
1.785	2.95	3
1.790	2.99	4
1.795	3.02	3
1.800	3.06	4
1.805	3.09	3
1.810	3.13	4

30. *Epoch Hour.*—This must now be assumed. It may be taken at any of the hours, but should be near the middle of the eclipse, which is not yet known, so we may take it near the time of conjunction. Although the Epoch Hour may be assumed anywhere, an *integral hour* must be taken, on account of the method here given for computing the hourly motions; and when once assumed, it cannot be changed without recomputing the *mean* hourly motions. The 9th hour is assumed in the present eclipse, and for this time the hourly motions of x and y are to be gotten in the Eclipse Tables.

The Epoch Hour is the standard from which all the computed times depend. We must, therefore, know the changes of x and y, counted *from this hour.* For example, the change between 9 and 10 hours is the average rate, which would be the change at 9.30. Likewise the change from 9^h to 11^h would be the mean or average change which is the mean of the two differences at 9.30 and 10.30. Also the change from 9^h to 12^h is one-third of the three differences. Thus, between 9^h and any other time these are seen to be the *average* or *mean* motions, and hence properly called by CHAUVENET, p. 451, line 3, the *Mean* Hourly Changes. It would be well to take for these the notation x_0' y_0'. They are used by the computer of the eclipse throughout, and are the quantities given by CHAUVENET on page 455.

The formula in this case is that for finding a first difference for any given argument (CHAUVENET'S *Astronomy,* I. 90):

$$f'T = \frac{1}{w}(\Delta_1 - \tfrac{1}{6}\Delta_3 + \tfrac{1}{30}\Delta_5 - \text{etc.}).$$

Here w is the fraction of the interval, which is unity in our case; we may omit the third term; Δ_1 and Δ_3 are to be taken as mean value of the two adjacent differences; this half sum is usually expressed by $\frac{1}{2}\Sigma\Delta$. Hence, for our use we have for the mean hourly changes—

For the epoch hour,

$$x_0' \text{ or } y_0' = \tfrac{1}{2}\Sigma\Delta_1 - \tfrac{1}{12}\Sigma\Delta_3. \tag{28}$$

And for the other hours, the second differences not being used,

At 6^h	$\frac{1}{3}$ of the three	differences	below it.
7^h	$\frac{1}{2}$ " two	"	"
8^h	the first	"	"
10^h	"	"	above it.
11^h	$\frac{1}{2}$ of the two	"	"
12^h	$\frac{1}{3}$ " three	"	"

On the other hand, the astronomer who wishes to observe the eclipse and compute the times beforehand knows nothing of our Epoch Hour, but assumes some hour near the time when the eclipse begins at his station, say 7^h, and another observer 11^h, according to his location, and they want to know the *actual* change *at* the hours they select. These we may term the *absolute* hourly changes, which are given by CHAUVENET on page 464, and are also the quantities given in the BESSELIAN Elements in the *Nautical Almanac,* after

being divided by 60 to give the changes in one minute. This is taking $w = 60$ in CHAUVENET'S formula above quoted.

The absolute hourly motions, which may be noted at x' y', are computed for *each* hour by the formula (28); and for this the differences of x and y will generally have to be extended above and below, as in the example. The numbers should all here be carried out to six decimals; though the last is not very accurate, yet it had better be retained. Get the logarithms of these to five places; these are omitted from the printed page for lack of room.

31. The following are called the constants for cones for eclipses:

$$\text{Penumbra } C = \sin H + k \sin \pi_0. \tag{29}$$
$$\text{Umbra } C_1 = \sin H - k \sin \pi_0. \tag{30}$$

They should be carefully computed with seven places of logarithms and written on the lower edge of a slip of paper for use in all eclipses. They are changed only when more accurate values of the constants entering are obtained.

$$\text{For Penumbra, } \sin f = \frac{C}{r'g}, \tag{31}$$
$$c = z + \frac{k}{\sin f}, \tag{32}$$
$$i = \tan f, \tag{33}$$
$$l = ic. \tag{34}$$
$$\text{For Umbra, } \sin f_1 = \frac{C_1}{r'g}, \tag{35}$$
$$c_1 = z - \frac{k}{\sin f_1}, \tag{36}$$
$$i_1 = \tan f_1, \tag{37}$$
$$l_1 = i_1 c_1. \tag{38}$$

In these formulæ f is the angle of the cone of shadow, l and l_1 the radii of the Penumbra and shadow on the fundamental plane; z, the distance of the moon's centre; $\frac{k}{\sin f}$, the distance of the vertex of the cone of shadow from the moon; and therefore c, the distance of the vertex of the cone *above* the fundamental plane.

Hence, we have the species of the eclipse:

When the sign of l_1 for Umbra is positive, the eclipse is annular.
" " l_1 " is negative, " is total.

l for Penumbra is always positive.

Tan f can readily be found from the sine by the annexed table, which gives log sec i for the corresponding values of log sine.

If addition and subtraction logarithms are used, *log — log* for Umbra is found at once from that of Penumbra, and is

log sin f.	log sec f.
7.6603 — 48	0.0000046
7.6649 — 95	47
7.6696 — * 41	48
7.6742 — 88	0.0000049

Greater than for Penumbra when the eclipse is total,

Less than for Penumbra when the eclipse is annular,

in each case differing by the difference of the constants for cones. Both formulæ should be computed, using this as a check.

Tabulate log l and log l_1 to six places of logarithms and pass to numbers, preferably six places. Tabulate log i log i_1 to five places of logarithms, and for these latter the natural numbers are not needed. Difference them all.

32. Compute the remainder of the tables with five-place logarithms and to seconds of arc.

$$\left.\begin{aligned}\rho_1 \sin d_1 &= \sin d,\\ \rho_1 \cos d_1 &= \cos d\sqrt{1-e^2},\end{aligned}\right\} \tag{39}$$

These quantities are used to take account of the compression of the earth instead of using d and ρ.

In equations of this form, in the first member the first factor is a linear quantity, and ALWAYS positive; the second factor is an angle, of which the signs are determined from the second member, and hence the quadrant. In the present case d_1 differs but little from the declination of the sun. Tabulate these quantities and difference them; log cos d_1 should also be tabulated, since it is frequently used in the succeeding calculations. Log sin d_1 we will pass over for the present.

$$b' = -y' + \mu'_1 x \sin d \tag{40}$$

$$\text{Penumbra} \quad c' = x' + \mu'_1 y \sin d + \mu'_1 il \cos d \tag{41}$$

$$\text{Umbra} \quad c'_1 = x' + \mu'_1 y \sin d. \tag{42}$$

The *Absolute* Hourly Motions are to be used here for the first terms. The only difference between c' for Umbra and Penumbra is in the third term, which for Umbra, on account of the small value of l, is almost or quite insignificant, and is omitted. This term for Penumbra is nearly constant for one eclipse, and varies from about 0.000610 to 0.000680, and this affects the angle E (next to be com-

puted) some 10′ or 15′ of arc; and yet Chauvenet computes E for Penumbra only and uses that for the limits of Total Eclipse. It is, however, in a formula where it has little effect. But the better way is to compute c' and E for the Umbra, and use them for the Penumbra, where less accuracy is necessary, so that equation (41) above may be wholly neglected, and (42) substituted in its place.

Compute the second terms and add by natural numbers. Transfer the numbers of b' to the tables and also get the logarithm there. c' is not needed among the tables, but may be placed there and differenced, or differenced on the line below in the computation.

The hourly motions of the above quantities are simply the logarithms of the mean of adjacent differences at the epoch hour, one value of each given to *four* decimals of logarithms.

$$b_0'' = \tfrac{1}{2}\Sigma\Delta_1 b'. \tag{43}$$

$$c_0'' = \tfrac{1}{2}\Sigma\Delta_1 c'. \tag{44}$$

The quantities b' and c' are the relative motions of the surface of the earth and the shadow; and the angle E next to be computed is the angle of this resultant path,

$$\left.\begin{aligned} e \sin E &= b' \\ e \cos E &= c' \end{aligned}\right\} \tag{45}$$

E may be tabulated to the tenth of a minute and differenced. And in equations of this form that are important I generally compute the linear quantity by both equations. In these equations I get first $\log e$, and then $\log \frac{1}{e}$, and tabulate the latter, because e is used only in the denominator. The natural numbers of e are not required.

33. Finally, to take account of the compression of the earth:

$$y_1 = \frac{y}{\rho_1}. \tag{46}$$

$$y'_1 = \frac{y'_0}{\rho_1}. \tag{47}$$

The former of these can be found when required, but the latter should be computed from the Mean Hourly Changes of y in the tables.

34. The following formulæ are given in Chauvenet; but are not generally required, and need not be computed.

Classed with b' and c' we have for Penumbra and Umbra:

$$a' = -l' - \mu' ix \cos d \tag{48}$$

Classed with μ',

$$d' = \frac{dd}{dT} \sin 1'' \tag{49}$$

Classed with E and e,

$$\left.\begin{aligned} f \sin F &= d' \\ f \cos F &= \mu' \cos d \end{aligned}\right\} \tag{50}$$

Classed with d and $\log \rho_1$,

$$\left.\begin{aligned} \rho_2 \sin d_2 &= \sin d \sqrt{1 - e^2} \\ \rho_2 \cos d_2 &= \cos d \end{aligned}\right\} \tag{51}$$

35. The quantities in the tables which vary rapidly must now be interpolated to every 10 minutes. The interpolation is generally simple, and need not be done here in this example. The quantities are as follows:

x, y, l, l, each to five places, natural numbers.
μ_1 and d_1 to seconds.
Log sin d_1 from the computation to five places of logarithms.
Log b' to four places of logarithms if possible. If not, then the natural numbers to five places.
E to tenths of a minute.
Log $\frac{1}{e}$ to five places of decimals.

In a computation like the present, where signs change frequently, it is a good plan to write the + sign, as well as the —, before each quantity at the head of a line or column. But in the middle of a line or column the sign may be omitted, except where they change, and also the characteristics and other figures which do not change. In using addition and subtraction logarithms, $l - l$ is the numerical difference of the logarithms; which is called A in addition, and B in subtraction; and the other letter, B or A, is then taken out from the table.

REMARK.—The author wishes it understood throughout this work that constants and other quantities which are repeated in the formulæ *are not given or repeated in the examples.* They should be written on a slip of paper, and used when required, as suggested in Article 20. There are two species of constants—those which are absolutely constant, and those which are constant for one eclipse.

36. The Main Computation for Tables.

Total Eclipse, 1904, September 9.

Formulæ.		7^h.	$T_0 = 9^h$.	12^h.
(15)	$\sin \pi$	+ 8.2517542	8.2517475	+ 8.2517063
	$r' \sin \pi$	+ 8.25470	8.25468	+ 8.25462
	$\log b$	+ 7.37536	7.37538	+ 7.37542
	$l - l$, B	2.62464	2.62462	2.62456
(16)	$A = \log (1:1 - b)$	+ 0.0010320	0.0010320	+ 0.0010321
(17)	$\cos \delta$	+ 9.9980500	.9983165	+ 9.9986816
	$\sec \delta'$	+ 0.00185	.00182	+ 0.00179
	$\log (\alpha - \alpha')$	— 3.57512	+ 2.55348	+ 3.81445
$-(a - \alpha') =$ sum,	log	— 0.95141	+ 9.93003	+ 1.19139
(18)	$\log \delta - \delta'$	+ 2.72552	— 2.86695	— 3.42363
$-(d - \delta') =$ sum,	log	+ 0.10191	— 0.24336	— 0.80010
(23)	$\log r$	+ 1.7482458	+ 1.7482525	+ 1.7482937
(24) (Check Eq. 27)	$\sin (\alpha - a)$	— 8.2616996	+ 7.2400759	+ 8.5009816
x is 0 at ☌.	$\log x$	— 0.0079954	+ 8.9866449	+ 0.2479569
(25)	$\sin (\delta - d)$	+ 7.4121222	— 7.5535539	— 8.1102273
	$\cos^2 \frac{1}{2}(\alpha - a)$	+ 9.9999638	+ 9.9999996	+ 9.9998909
Log r is given above.	log (1)	+ 9.1603318	— 9.3018060	— 9.8584119
	$\sin (\delta + d)$	+ 9.26884	+ 9.25189	+ 9.22492
	$\sin^2 \frac{1}{2}(\alpha - a)$	+ 5.92138	+ 3.87806	+ 6.40001
	log (2)	+ 6.93847	+ 4.87820	+ 7.37322
	$l - l$,	A 2.22186	B 4.42361	B 2.48519
	B or A	B + 0.0025979	A — 0.0010164	A — 0.0014233
	$\log y$	+ 9.1629297	— 9.3017896	— 9.8569886
(26) All the terms of	$\cos (\delta - d)$	+ 9.9999986	9.9999972	+ 9.9999639
this equation are	log (3)	+ 1.7482082	.7482493	+ 1.7481485
positive always.	$\cos (\delta + d)$	+ 9.99238	.99296	+ 9.99379
Log r in this formula is given above.	log (4)	+ 7.66201	5.61927	+ 8.14209
$l - l$ always B for Z.	$l - l$, B	4.08620	6.92898	3.60606
	A	0.0000356	.0000003	0.0001076
Log Z is always +. Natural numbers are not required.	$\log Z$	+ 1.7481726	.7482490	+ 1.7480409
(31)	$\log (1 : r'g)$	+ 9.9980898	.9980992	+ 9.9981135
(Constant, Eq. 29).	$\sin f$	+ 7.6668507	.6668601	+ 7.6668744
(32)	$(k : \sin f)$	+ 1.7685253	.7685159	+ 1.7685016
	$l - l$, A	0.0203527	.0202669	0.0204607
	B	0.2909729	.2910147	0.2909201
	$\log c$	+ 2.0594982	.0595306	+ 2.0594217
(33)	$\log i$	+ 7.6668554	.6668648	+ 7.6668791
(34) For Penumbra l is always +.	$\log l$	+ 9.7263536	.7263954	+ 9.7263008
(35) (Const't, Eq. 30.)	$\sin f_1$	+ 7.6646807	.6646901	+ 7.6647044
(36)	$(k : \sin f_1)$	+ 1.7706953	.7706859	+ 1.7706716
	$l - l$, B	0.0225227	.0224369	0.0226307
	A	1.2963765	.2979916	1.2943525
	$\log c_1$	— 0.4743188	.4726943	— 0.4763191
(37)	$\log i_1$	+ 7.6646854	.6646948	+ 7.6647091
(38) *Total Eclipse.*	$\log l_1$	— 8.1390042	— .1373891	— 8.1410282

Art. 37. Eclipse Tables for Total Eclipse, 1904, September 9.

G. M. T.	$+(a-a')$.		(18) a.		$(\alpha-a)$.				$\frac{1}{2}(\alpha-a)$.
6^h	+ 13.84	+ 4.90	167° 40′ 13″.17	+ 2 10.11	− 1° 37′ 13″.92	− 34 25.59	− 1.49	+ 13	− 0 48 36.96
7	8.94	4.90	42 23 .28	10	1 2 48 .33	34 24.10	1.36	+ 2	31 24.16
8	+ 4.04	4.89	44 33 .38	11	− 0 28 24 .23	34 22.74	1.34	− 1	− 14 12.11
9	− 0.85	4.90	46 43 .49	10	+ 5 58 .51	34 21.40	1.35	− 14	+ 2 59.25
10	5.75	4.89	48 53 .59	10	40 19 .91	34 20.05	− 1.49		20 9.95
11	10.64	+ 4.89	51 3 .69	+ 2 10.09	1 14 39 .96	34 18.56			37 19.98
12	15.57		167 53 13 .78		+ 1 48 58 .52				+ 0 54 29.26

$\Theta + h$.

168° 6′ 14″.70

		(19) μ_1.	
6^h	258° 21′ 1″.8	90° 40′ 48″.6	+ 15° 0′ 17″.7
7	273 23 29 .6	105 41 6 .3	17 .8
8	288 25 57 .5	120 41 24 .1	17 .7
9	303 28 25 .3	135 41 41 .8	17 .8
10	318 30 53 .2	150 41 59 .6	17 .7
11	333 33 21 .0	165 42 17 .3	+ 15 0 17 .8
12	348 35 48 .9	180 42 35 .1	

(20) $\log \mu_1$ + 9.418111

$\cos d$ + 9.99818

(22) $\log f$ + 9.4163

(21) $\log \mu'$ for *N. A.* + 1.1762

	$+(d-\delta')$.		d.		$(\delta-d)$.				$(\delta+d)$.
6	− 2.76	+ 1.50	+ 5° 17′ 33″.84	− 55.22	+ 19′ 24″.86	− 10 32.08	− 2.20	+ 11	+ 10° 54′ 32″
7	− 1.26	1.50	16 38 .62	55.22	+ 8 52 .78	34.28	2.09	8	42 10
8	+ 0.24	1.51	15 43 .40	55.23	− 1 41 .50	36.37	2.01	12	29 45
9	1.75	1.52	14 48 .17	55.22	12 17 .87	38.38	1.89	+ 1	17 18
10	3.27	1.52	13 52 .95	55.23	22 56 .25	40.37	− 1.88		10 4 50
11	4.79	+ 1.52	12 57 .72	− 55.25	33 36 .52	− 10 42.15			9 52 19
12	+ 6.31		+ 5 12 3 .47		44 18 .67				+ 9 46 46

	$\log x$ (28)	x.				(28) x_0'.		$\log x_0'$ (28).	x'.	
			639							
6	− 0.1976362	− 1.576290	+ .557709	70		+ 0.557754	+ 23	+ 9.74644	+ 0.557678	+ 61
7	.0079954	.018581	.557762	+ 33	− 24	777	+ 14	46	739	43
8	− 9.6635307	− 0.460819	.557791	+ 29	54	791	− 4	47	782	+ 5
T_0 9	+ 8.9866449	+ .096972	.557766	− 25	56	.557787	21	47	787	− 52
10	9.8160693	.654740	.557685	81	− 96	766	41	46	735	129
11	0.0836550	1.212425	+ .557508	− 177		725	− 72	42	606	− 190
12	+ 0.2479569	+ 1.769933	308	200		+ 0.557653		+ 9.74636	+ 0.557416	

In this table the numbers in parentheses denote the formulæ by which the quantities are computed.

T_0	log y.	y.				(28) y'_0.		log y'_0.		(28) y'.	
6	+ 9.5029094	+ 0.318353	— .172831	100		— 0.172901	— 35	— 9.23779	— 9	— 0.172785	— 88
7	+ .1629297	+ .145522	.172908	— 77	+ 21	36	28	788	8	873	66
8	— 8.4375280	— .027386	.172904	56	13	64	25	796	6	939	50
9	9.3017896	.200350	.173007	43	+ 26	.172989	18	802	5	989	28
10	.5721238	.373357	.173024	17	— 8	3007	8	807	2	.173017	17
11	.7374957	.546381	— .173049	— 25		3015	— 12	809	— 3	634	— 25
12	— 9.8569886	+ 0.719430		30		0.173027		— 9.23812		— 0.173059	

	log l.			l.			log i.	log l_1.		l_1.		log i_1.
6	+ 9.726301	+ 53		+ 0.532477	+ 65		+ 7.66685	— 8.14102	+ 202	— 0.013836	+ 64	+ 7.66468
7	354	31	— 22	542	38	— 27	86	.13900	121	772	39	69
8	385	+ 10	21	580	+ 13	25	86	.13779	+ 40	733	+ 10	69
9	395	— 10	20	593	— 13	26	86	.13739	— 41	723	— 10	69
10	385	31	21	580	38	25	87	.13779	121	733	29	70
11	354	— 55	— 24	542	— 65	— 27	87	.13901	— 202	772	— 64	70
12	+ 9.726301			+ 0.532477			+ 7.66688	— 8.14103		— 0.013836		+ 7.66471

	d_1.		log cos d.	log ρ_1.	b.			log b'.		
6	+ 5° 18 38	— 55	+ 9.99813	+ 9.99854	+ 0.134706	+ 13629		+ 9.12939	+ 4185	
7	17 43	56	14	54	.148335	13536	— 93	.17124	3793	— 392
8	16 47	55	15	55	.167871	13440	96	.20917	3464	329
9	15 56	56	16	55	.175311	13340	100	.24381	3185	279
10	14 56	55	17	55	.188651	13248	92	.27566	2948	237
11	14 1	— 56	18	55	.201899	+ 13175	— 73	.30514	+ 2745	— 203
12	+ 5 13 5		+ 9.99820	+ 9.99854	+ 0.215074			+ 9.33259		

	c' for Umbra.		E for Umbra.			log $\frac{1}{e}$ for Umbra.			log y_1 (47).	
6	+ 0.565368	— 4124	+ 13° 24.1	+ 1 24.2		+ 0.23568	+ 51		— 9.23925	— 9
7	1244	4120	14 48.3	1 23.8	— 0.4	619	26	— 25	934	8
8	.557124	4134	16 12.1	1 23.3	.5	645	+ 3	23	942	6
9	2990	4170	17 35.4	1 22.8	.5	648	— 15	22	948	5
10	.548820	4223	18 58.2	1 22.3	.5	633	37	22	953	2
11	1597	— 4259	20 20.5	+ 1 21.8	— 0.5	596	— 56	— 19	955	— 3
12	+ 0.540338		+ 21 42.3			+ 0.23540			— 9.23958	

(43).

b'' + 0.013390
log + 8.1268

(44).

c'' — 0.004152
log — 7.6182

ART. 36. Computation. (*Continued.*)

(39)	$\sin d$	+ 8.96368	8.96116	+ 8.95732
	$\cos d$	+ 9.99815	.99818	+ 9.99821
	$\cos d \sqrt{\ }$	+ 9.99668	.99671	+ 9.99674
	$\tan d_1$	+ 8.96700	.96445	8.96060
	d_1	+ 5 17 43	5 15 52	+ 5 13 5
	$\sin d_1$	+ 8.96514	8.96261	8.95871
		+ 9.99854	855	9.99855
	$\cos d_1$	+ 9.99814	9.99816	9.99820
Log ρ_1 is always +.	$\log \rho_1$	+ 9.99854	9.99855	+ 9.99854
(40)	$\mu_1 x \sin d$	— 8.38983	+ 7.36591	+ 8.62340
y' is found in the	Numbers	— 0.024538	+ .002322	+ 0.042015
Eclipse Tables	b'	+ 0.148335	+ .175311	+ 0.215074
(42)	$\mu_1 y \sin d$	+ 7.54472	7.68106	— 8.23244
x' is found in the Eclipse Tables.	Numbers	+ 0.003505	.004797	— 0.017078
c' is always +.	c'	+ 0.561244	.552990	+ 0.540338
(45)	$\log b'$	+ 9.17124	.24381	+ 9.33259
	$\log c'$	+ 9.74915	.74272	+ 9.73266
	$\tan E$	+ 9.42209	.50109	9.59993
	E	+ 14 48 16	+ 17 35 23	+ 21 42 18
	$\sin E$	+ 9.40743	+ 9.48029	+ 9.56800
	$\log e$	9.76381	9.76352	9.76459
	$\cos E$	+ 9.98534	+ 9.97920	+ 9.96806
Log e is always +.	$\log (1 : e)$	+ 0.23619	+ 0.23648	+ 0.23540

The numbers of formulæ omitted above are given in the Tables, in which further but brief computations are made.

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38. *Geometrical Illustration of the Foregoing Quantities and the Eclipse Generally.*—The Fundamental Plane in the theory of eclipses to which all quantities are referred we will take as the plane of the paper. It is the plane XY of the coördinate axes which intersect in the centre of the earth. The radius of the earth is the unit of measure, and has been the unit of all the quantities and decimals computed for the Eclipse Tables.

Fig. 2, Plate I., is an orthographic projection of our example, the Total Eclipse of 1904, September 9. In this, as well as in the other principal figures of this work, the unit of measure is assumed to be two and one-half inches; a scale of 40 parts will then give hundredths of the scale, and we can estimate to three decimals.

From the centre, Z, of the coördinate axes draw the circle $ADBC$, representing the earth—the compression being neglected. The north pole of the earth is in the plane YZ, near the point C, its position to be determined presently. The point marked Z is the axis of Z, perpendicular to the plane of the paper. The letter Z

ART. 37. (*Continued.*)

A PORTION OF THE INTERPOLATION OF THE ECLIPSE TABLES FOR 10 MINUTES.

G. M. T.	x.	y.	l.	l_1.	μ_1.	d_1.	log sin d_1
6ʰ 0ᵐ	−1.57629	+0.31835	+0.53248	−0.01384	90° 40′ 49″	+5° 18′ 38″	+8.96640
10	1.48334	.28955	.53249	82	93 10 54	29	619
7 0	1.01858	.14552	.53254	− .01377	105 41 6	5 17 43	8.96514
10	0.92562	.11671	55	76	108 11 9	33	493
20	.83266	+ .08789	56	75	110 41 12	24	472
8 0	.46082	− .02739	.53258	− .01373	120 41 24	5 16 47	8.96387
10	.36786	.05621	58	73	123 11 27	37	366
20	.27490	.08503	58	73	125 41 30	28	345
30	.18194	.11386	58	72	128 11 32	19	324
40	− .08897	.14269	59	72	130 41 36	10	303
50	+ .00400	.17152	59	72	133 11 39	1	282
9 0	.09697	.20035	.53259	− .01372	135 41 42	5 15 52	8.96261
10	.18994	.22918	59	72	138 11 45	43	240
20	.28290	.25802	59	72	140 41 48	33	219
30	.37586	.28685	59	73	143 11 51	24	198
40	.46882	.31569	58	73	145 41 54	15	177
50	.56178	.34452	58	73	148 11 57	6	156
10 0	.65474	.37336	.53258	− .01373	150 42 0	5 14 56	8.96135
10	.74769	.40219	57	74	153 12 2	47	114
20	.84064	.43103	57	74	155 42 5	38	093
30	.93359	.45986	56	75	158 12 8	28	072
40	1.02654	.48870	56	75	160 42 11	19	050
50	1.11948	.51754	55	76	163 12 14	10	029
11 0	1.21242	.54638	.53254	− .01377	165 42 17	5 14 1	8.96007
10	1.30534	.57522	53	78	168 12 20	13 51	.95986
20	1.39826	.60406	52	79	170 42 23	42	965
30	+1.49118	−0.63290	+0.53251	−0.01380	173 12 26	+5 13 33	+8.95944

also sometimes denotes the centre of this circle; sometimes the surface of the earth above it; and sometimes the point in the celestial sphere which is the zenith of the projection; and in this respect is the point Z referred to in CHAUVENET'S *Astronomy*, vol. i., p. 441. These several points are all in the same straight line, and there will be no ambiguity in letting one letter represent them all. And Z being the zenith, the circle is the horizon at any time.

Next lay off the values of x along the axis of X—the negative values on the left, the positive on the right,* because the shadow

* In SCHILLEN'S *Spectrum Analysis*, p. 207, the following unaccountable collection of mistakes is made: "As, however, the moon, which throws the shadow, only completes its course in the heavens round the earth from west to east in one month, and the earth, which receives the shadow, accomplishes its revolution from west to east in one day, it follows that the motion of the moon's shadow is very much slower than that of the earth's surface. It therefore happens that the earth appears to run away from under the moon's shadow, or that the moon's shadow seems to run over the earth from East to West."—Translation by JANE and CAROLINE LASSELLE. Edited by WILLIAM HUGGENS. D. Appleton & Co.

passes from *west to east* over the earth; on these points erect the ordinates y, measuring the values of y already computed, and the line connecting these latter points is the path of the centre of the shadow across the fundamental plane. This line is a curve, but of very slight curvature. The hours are noted on the path. The axis of the shadow, which is the line through the centres of the sun and moon, is perpendicular to the plane of the paper at all times.

This shows the beauty of the modern methods of LAGRANGE, HANSEN, and BESSEL, that the right ascensions and declinations of the sun and moon are transformed into the right ascension and declination, a and d of the point Z; and from these the coördinate axes are computed, so that we now may consider an apparently fixed plane with the shadow moving across it. The earth can readily be referred to this plane.

39. *Principles of Spherical Projection.*—We will digress for a moment to recapitulate some of the fundamental principles applicable to the orthographic projection of the sphere. The primitive plane of the projection is here the Fundamental Plane, and its intersection with the earth, $ADBC$, the *Primitive Circle*, and CD, the Principal Meridian. All circles parallel to the primitive plane project as circles of their true size; all circles perpendicular to the plane project into right lines, as the line CD. Circles oblique to the plane project in ellipses, such as the earth's equator, which will be projected presently. We may revolve the whole sphere or only a portion of it. If we revolve it round the axis Z, 90°, A will fall at D, D at B, etc., and all points of the sphere will revolve through *the same angle*, though the distance passed over is *less* the nearer the point is to the axis of revolution. In any revolution all points will revolve in planes perpendicular to the axis of revolution.

Now revolve the sphere 90° about the line CD. The semicircle CAD and the fundamental plane will fall in the right line CD. The semicircle CD in space will fall in the semicircle CBD. In this position we know the position of the north pole and equator. The point Z of the celestial sphere will lie in the indefinite line $Z\Upsilon$, which in our example has the declination $d = +5° 16'$ at the time of conjunction; therefore, lay off the angle BZE' *below* ZB, so as to make the latter *north* declination, and ZE' will be a portion of the equator of the earth projected in a right line. It is a principle of this projection that the elevation of the pole equals the latitude of the point Z. Hence we may lay off CP'. P' is the north

pole of the earth, and $P'Z$, a part of the axis in its revolved position.

When the sphere is revolved back, P' will fall in the line CD at P, which is the pole of the earth in the projection. E' will fall at E, and the ellipse AEB is the equator projected obliquely.

40. Fig. 3, Plate II.—*Total Eclipse of 1860, July 18.*—This is the eclipse CHAUVENET has taken as his example, and the projection is made with his data as follows:

DATA FOR FIG. 3, TOTAL ECLIPSE, 1860, JULY 29.

G. M. T.	x	y	μ_1	E	Constants in Projection.	
0^h	-1.172	$+0.917$	358° 31′	4° 33′		
1	0.627	0.757	13 31	9 22	$l + 0.536$	$\mu'_1 + 0.262$
T_0 2	-0.081	0.596	28 31	11 17		
3	$+0.464$	0.435	43 31	19 14	$l_1 - 0.009$	$\sin d + 0.358$
4	1.009	0.274	58 31	24 8		
5	$+1.554$	$+0.112$	73 31	28 55	$d + 20°\ 57'$	$\mu' \sin d + 0.094$

1^h mean time, sidereal interval	$15^h\ 2^m\ 28^s$
Correction for equa. time and reduc. to a —	1 31 18
μ_1 at 1^h	13 31 10

The foregoing remarks, as well as much that follows, are applicable to both figures, in which the reference letters refer to similar parts. The two eclipses differ from each other considerably in some respects.

41. To resume the consideration of the tables, x_0' x' (for their difference is so small we cannot distinguish between them in the drawing) are the spaces between the ordinates of the path; likewise y_0' and y' are the differences of the consecutive ordinates; and shown in the right-angled triangles at H, y' is negative in each figure.

With the radius l for Penumbra, describe arcs of circles from each of the hour points on the path, which are the intersections of the penumbral cone with the fundamental plane. In Fig. 2 two of these curves are entire, showing that *all* the shadow falls upon the earth; in Fig. 3 it does not; and this circumstance causes the two principal forms of the rising and setting curve, seen in the *Nautical Almanac*, and shown in Fig. 17, Plate VI., and Fig. 18, Plate VII., of this work. The circles for the umbral cone are too small to be shown on the scale of Figs. 2 and 3, and are therefore omitted.

42. It was stated in Article 39 that the elevation of the north pole

above the fundamental plane is equal to the latitude of the point Z, or the declination of the sun. There is, however, this important distinction: If we call φ the declination of the sun, and d the elevation of the north pole, the compression of the earth for these two points will not be the same. The compression for the angle of elevation d will be the compression for the latitude $\varphi = 90° - d$.

To understand the quantities d and $\log \rho$, the former must be considered as allied to the elevation of the pole and with no affinity to the declination of the sun. In Fig. 3 (the elevation in Fig. 2 being too small to show this well) d is the angle of elevation of the pole, measured in a vertical plane; but when revolved down in the fundamental plane, is the angle $P'ZP$.

It should first be remarked that analytical geometry gives the minor axis of an ellipse, $b = a\sqrt{1 - e^2}$, e being the eccentricity; and in the present case a, the equatored radius of the earth, and major axis, is taken as unity. We also learn that if a circle is described upon the "major axis of an ellipse, all the ordinates of the ellipse will be shortened from those of the circle in the same ratio as the minor axis"; that is, in the ratio $\sqrt{1 - e^2}$.

Now sines and cosines are always given to a radius unity; when $\cos d$ is multiplied by this quantity it will very nearly represent the cosine of the arc of the sphere at C. CZ, it will be remembered, is less than unity on account of the compression, and $P'Z$ is still less on account of the compression being greatest at the pole.

If we take the sum of the squares of equation (39), which gives $\log \rho_1$, we get, after reducing, $\rho_1 = \sqrt{1 - e^2 \cos^2 d}$. The general expression for the radius of the earth is,

$$\rho = \sqrt{\frac{1 - 2e^2 \sin^2 \varphi + e^4 \sin^2 \varphi}{1 - e^2 \sin^2 \varphi}},$$

in which, if we make a small approximation, writing $\sin^4 \varphi$ for $\sin^2 \varphi$ in the last term of the numerator, the latter becomes a perfect square, and we have

$$\rho = \sqrt{1 - e^2 \sin^2 \varphi}.$$

This becomes the same precisely as the former equation, if we make $\varphi = 90 - d$, or, what is the same thing, write $\cos d$ for $\sin \varphi$, so that the quantity ρ_1 is very nearly represented by the line CZ (Fig. 3).

We can now understand the office which $\log \rho_1$ fulfils. It results directly from CHAUVENET's transformation of the geographical latitude φ through the geocentric φ' to the transformed latitude φ_1. ρ_1

appears as a divisor to the coördinate η, which is placed equal to η_1; thus,

$$\eta_1 = \frac{\eta}{\rho_1}$$

and ζ_1 assumed, so that

$$\xi + \eta_1 + \zeta_1 = \text{unity},$$

the meaning of which is that the earth's spheroid is replaced by a sphere (radius *unity*) which lies outside of the earth at the poles, because $\eta_1 > \eta$; and probably within the earth at the equator. And in order to retain the proper distance between the earth and shadow path, its ordinate y is likewise divided by ρ_1, giving y_1.

43. The Greenwich hour angle of the principal meridian *PD* is μ_1, which is shown clearly thus:

$$\mu_1 = \theta + h' - a = \theta + h' - (a' + \text{small term, equa. 17}).$$

But

$$\theta - a' = -\text{ equation of time.}$$

Hence,

$$\mu_1 = h - \text{equa. time} - \text{small term.} \tag{52}$$

As the hour angle is counted from *apparent* noon, we must reduce from *mean* noon to *apparent* by applying the equation of time according to its sign on the mean time page of the *Almanac*, which is its negative value. The small term reduces the angle from the sun's right ascension to the quantity a. We will compute this for 9^h by both formulæ for comparison:

θ	168° 6′ 14″.70		— Equa. time		+ 2^m 46^s.73
h	135 0 0		— " " arc		0° 41′ 40″.95
Reduc. to sid.	0 22 10 .62		h mean time		135 0 0
	303 28 25 .32		Small term — (— 0.86)		+ 0 .86
a	167 46 43 .49		μ_1		135 41 41 .81
μ_1	135 41 41 .83				
			Reduc. to sid.		0° 22′ 10″.62
a'	167° 46′ 44″.34		$\theta - a'$		0 19 30 .36
$\theta - a'$	0 19 30 .36		Equa. time		0 41 40 .98

It is seen that the equation of time takes the place of the two smaller terms, being just equal to them.

In the first formula h' is in sidereal time, and in the second formula h is in mean time.

Now suppose in Fig. 2 the earth be revolved so that the north pole, P, falls in the point Z, the circle $ADBC$ will be the equator and ZD the principal meridian at all times while the earth revolves. We will now set off from D, the hour angle of $9^h = 135°$, toward the

right, to the point g, which is Greenwich *mean* noon; but the meridian of Greenwich is not yet overhead until we further set off the equation of time to Greenwich *apparent* noon, G. This and the small reduction to a, the meridian of the point Z, gives us the hour angle μ_1, between the Greenwich meridian and the principal meridian, PZD. We may likewise set off the other hours from 6 to 12, which give the hour angles at these hours.

44. Lastly, we have the quantities b', c', E, and e to illustrate; they are closely related.

If we omit the small terms, we have—

$$\tan E = \frac{b'}{c'} = \frac{-y'}{x'} \tag{53}$$

As E is counted from the axis Y, set off x' from Z (Fig. 2) along the axis, and $-y'$ (which is positive) on the right to the point F. This value of E gives a line at *right angles* to the path. The small terms $\mu'_1\, x \sin d$ and $\mu'_1\, y \sin d$ may be represented in the following manner: μ'_1 in natural numbers is 0.262, and $\sin d$ 0.091; and x, being multiplied by these, is reduced by the amount of their product, 0.024; so that x and y, if drawn on a reduced scale of 0.024, will represent these small terms. This is too small in Fig. 2 to be shown on account of the small declination, but in Fig. 3 we have from the data, Art. 40, μ'_1 0.262, $\sin d$ 0.358 and their product, 0.094, the reduction for the miniature representation of the path. Now transfer this figure parallel to itself, so that the point Z is placed upon F, and number the hours. By this means we add the small terms to the large one, giving b', the several distances of the points 1, 2, 3, etc., from the axis CD; and c' their distances from the axis AB. We also have the angle E at 1^h, $1\,ZC$, and e the distance $1\,Z$.

E at the middle of the eclipse gives a line very nearly at right angles to the path of the shadow. We shall recur again to this angle under Art. 97, on the Northern and Southern Limiting Curves.

45. *Motion of the Successive Eclipses of a Series.*—Applying the Criterion in Art. 12 to the eclipse of 1904, September 9, we see that it is at the moon's ascending node, and the series is therefore moving south. As the eclipse at local apparent noon, the point where the path crosses the axis of Y, is south of the point Z, the successive eclipses are decreasing. In Fig. 2, Plate I., the path of the preceding eclipse of 1886 is shown. The eclipse of 1860, July 29 (Fig. 3, Plate II.), CHAUVENET'S example, is at the moon's descending node, the series moving north, and successive eclipses decreasing. The

paths of the two following eclipses, 1878 and 1896, are also shown in this figure. And here we notice another point, the inclination of the path is changed at each appearance. When the angle gets to its maximum, it will then decrease, and in time the path, instead of moving south across the earth, will move north.

To ascertain whether the inclination of the path will increase or decrease for the next eclipse, we have as follows: The Saros (Art. 11) is 18^y 10 or $11^d\ 7^h\ 42^m$. Our year is the period of the sun's revolution, and the time is sure to be longer than the year by 10 or 11 days, in which time the sun's longitude has increased some 10 degrees, and its right ascension between the limits of 36^m and 44^m. As the sun and moon will be together at the next eclipse, the moon's right ascension must also increase by the same amount. Taking the eclipse Sept. 9, 1904, 9^h the time of conjunction, we have from the moon's ephemeris for this date and hour the

Change between 9 and 10 hours,	$+2^m\ 26^s$	and	$-11'\ 34''$
And on page 146 for the sun,	9		$-\ \ 57$
	137		$-\ \ 637$

The angle of the path will be 637 divided by 137, when the latter is reduced to arc and multiplied by the moon's declination, but as it is only the *relative* values of the angle we want, we may dispense with this reduction, and the quotient of the above is 46.5.

Next take the sun's right ascension ten days later, $11^h\ 45^m\ 40^s$. In the moon's ephemeris this value occurs on the 9th at 23^h:

The hourly motions at the time are	2 25	$-11\ 52$
Sun's hourly motions Sept. 19,	9	$-\ \ 58$
Difference,	136	654
The quotient is		48.1

Comparing this with the previous quotient, we see that the inclination is slightly increasing. The method is only a rough approximation, but it is sufficient for the purposes for which it is intended, in connection with Table XV. at the end of this work. We see in Fig. 2, Plate I., that the inclination has increased in 1904 since 1886.

The motion of the successive eclipses north or south must be estimated on the axis of Y, for this is the time of conjunction. In Fig. 3, Plate II., the total eclipses of 1860, 1878, 1896, it is seen that after two more appearances, the central path will fail to touch the earth, and the eclipse will be *partial*. About twelve partial eclipses will then occur, when the shadow will fail to touch the earth and the series will have run out. The series commenced at the south pole, followed by 12 or 13 partial eclipses; then 40 or 50 total, the

whole series of any eclipse, Professor NEWCOMB states, occupies about 1000 or 1200 years, during which there are 60 or 70 eclipses. The series of 1860, it is seen, passed over about 0.10 of the earth's radius in 18 years, which is greater than usual. The eclipse of 1883–1901, May, passed over 0.08, while that of 1883–1901, October, passed over but 0.03. The motion is variable in one series. Our example, 1904, September 9, has passed over since 1886, 0.06, as shown in Fig. 2.

46. *The Umbral Cone and Species of Eclipse.*—If in equation 26 we omit the small term and place the cosines of small angles equal to unity, we have

$$z = r = \frac{1}{\sin \pi},$$

and with the values of the moon's parallax given in Art. 10, we have the greatest, least, and mean values of the moon's distance, z, as given in the table below.

The greatest, least, and mean values of the sun's distance, as given by its logarithms in NEWCOMB's tables of the sun, and corrected for the constant 0.0000460 there deducted, are

0.0072147 9.9926636 0.0000000

Computing the umbral cone for these four maximum and minimum values, there results the several quantities given in the subjoined table:

THE UMBRAL CONE.

Moon's Parallax.	Sun's Distance. Greatest. $\log r'$ 0.00721		Least. $\log r' = 9.99266$	
Greatest. 61′ 28.8	$z =$	55.92	z	55.92
	$\frac{k}{\sin f}$	59.50	$\frac{k}{\sin f}$	57.54
	c	− 3.58		− 1.62
	l_1	− 0.016	l_1	− 0.008
Least. 53′ 55″.0	z	63.76	z	63.76
	$\frac{k}{\sin f}$	59.50	$\frac{k}{\sin f}$	57.54
	c	+ 4.26		+ 6.22
	l_1	+ 0.019	l	+ 0.029
Mean. 57′ 2.55	z	60.27	z	60.27

We will now plot these quantities in a diagram (Fig. 4, Plate III.) to a vertical scale of half an inch to one unit, which is the earth's radius, and assume the horizontal line AB as the mean fundamental plane to which all other quantities will be referred. The distance z is the distance of the fundamental plane from the moon. Drawing lines for the greatest and least distances and also computing these points for each minute, we have on the left of the figure the quantities depending upon the moon's distance. This determines the position of the fundamental plane, which depends chiefly upon the moon's parallax.

The values of $\frac{k}{\sin f}$ computed for the sun's greatest distance are very nearly the same—59.513 and 59.494; and for the sun's least distance, 57.548 and 57.529, differing only two units in the second decimal. This shows that the moon's parallax has but little effect in determining the position of the vertex of the umbral cone. Mean values are, however, given in the above table, and thence c, by which the vertex of the cone can be located thus: The quantity c is 3.58 below the upper extreme plane, and 4.26 above the lower extreme; their numerical sum is 7.84, the distance of the two extreme planes apart. Likewise, for the sun's least distance, we have -1.62 and $+6.22$, whose numerical sum is also the same. These give the vertices of the umbral cone in its extreme positions, which points we can set off, drawing horizontal lines from them to the right side of the drawing. Between these two latter lines, if we describe a circle, the tangent point to the lower line, the sun's least distance may be taken as the point of solar perigee at (nearly) January 1; the upper tangent point will be the apogee, July 1, and pointing off the months on this little circle we may thus very nearly ascertain the position of the vertex of the umbral cone at any time of the year; for it is almost independent of the moon's parallax. The moon's parallax being given, we can at any time by it locate the position of the fundamental plane. Just above the vertex of the cone on the line representing the moon's greatest parallax lay off the radii of the umbral cone -0.016 and -0.008. Likewise on the line of least parallax lay off the other values of the umbral cone, $+0.019$ and $+0.029$. Lines from these points to the two vertices already located will form the elements of two cones superimposed; and the points taken three and three will be found to be nearly in the same straight line. These values, which are

radii of the cones are here laid out as diameters, so as to enlarge the scale of measurement.

In the present eclipse the data September 9 and parallax 61′ 23″ give the radius of shadow for total eclipse, 0.0137, which is substantially correct, the construction being shown in dotted lines as follows: From the given date, September 9, draw the horizontal line to *v*, which is the vertex of the cone; draw the other element, *ve*. Then draw from the given value of the moon's parallax, 61′ 23″, the horizontal line *ae*, which will represent the Fundamental Plane and its intersection, *be*, with the cone, is the radius of the shadow, which can be measured by scale. This figure may be used for any eclipse.

CHAUVENET'S example, July 29, parallax 59′ 48″, gives also a total eclipse and radius of cone, 0.009.

47. But the diagram tells us much more than this. We see that as the mean fundamental plane lies in the shadow of an *annular* eclipse, the shadows of these must be generally greater in diameter than in total eclipses; which is true. Also, as the moon vibrates between the extreme parallaxes, if it is near one limit at one eclipse it will very likely be near the other limit for the other eclipse six months after; and both eclipses will be large. This occurs notably in 1901, 1904, etc. If, on the other hand, the moon is not near its limit at one eclipse, it will very likely not be near either limit six months after, and both eclipses will be small. It may happen that in both cases the moon is near the mean parallax, and two annular eclipses will occur, which was the case in 1897, when there were only two eclipses, both solar and both annular. Sometimes when the sun is not near the node there may be two partial eclipses at one node and one at the other node. In 1888 there were three partial eclipses, *all* of the annular species, and no total solar that year.

If in the mean fundamental plane we draw a semicircle representing the earth's hemisphere, we see that the vertex of the lower cone is within the earth's surface. In this case, as the shadow moves over the earth, at the beginning and end, the eclipse will be annular and total in the middle. This occurred in 1890, December 11, and it is properly called a Total-Annular Eclipse. It should be classed as Annular because l_1 for umbra has the positive sign. The moon's parallax for this eclipse is 59′ 8.″2 at conjunction.

48. From this diagram it seems at first glance as though we might approximate in numbers to the relative frequency of total and annu-

lar eclipses; but I do not think it can be done. The vertex of the cone moves quite regularly between its extremes; but the moon does not. Its motion as to its parallax is very irregular, seldom reaching the extremes here given. Another fact which would vitiate any computation is that at either node there may be one or two solar eclipses. After a series of years in which there is but one, another series may enter, making two. The best method of arriving at the proportion is to count the number of eclipses in one Saros, counting the partial eclipses as total or annular, according to the sign of l_1. Total-Annular Eclipses should be classed as Annular on account of the sign of l_1 in the Fundamental Plane. And here is a third circumstance which would again vitiate any computation—a total eclipse may after lapse of years become annular, and *vice versâ*.

49. This fact does not seem to be stated in any general *Astronomy*, that the species of total or annular may change. In this connection the following successive eclipses of one series are interesting; the semidiameters only are given:

	Sun's Semidiameter.		Moon's Semidiameter.	
1800, Interpolated. *Brit. N. A.*	16′ 7″.3	+1.4	16′ 18″.	−1
1818, " " "	16 8 .7	2.1	16 17 .	−2.4
1836, Nov. 8, Total Eclipse. *Brit. N. A.*	16 10 .8	3.1	16 14 .6	−2.5
1854, Nov. 19, Total Eclipse. " "	16 13 .9	+2.1	16 12 .1 old value. 16 14 .7 Sd. inc. 2″.6	
1872, Dec. 29, Annular Eclipse, *Amer. Eph.*	16 16 .0	−1.0	16 8 .7	−6.0
1890, Dec. 11, Total Annular. "	16 15 .0		16 6 .1	2.6
1908, 1927, Total Annular. OPPOLZER, *Canon of Eclipse.*				
1945, Annular Eclipse. " " "				

The change in the English *Nautical Almanac* of 1854, increasing BURCKHARDT'S value of the moon's semidiameter by 2″.60, it is curious to note, changed the *species of the eclipse.* With the old value the eclipse would have been Total-Annular. In 1872 it seems doubtful whether the true character of the eclipse was known to the computer, since the limits of the umbral cone were not required in those days, and hence the change in the sign of L on the earth's surface escaped notice. It is classed as an Annular Eclipse in the *American Ephemeris.*

SECTION V.

EXTREME TIMES GENERALLY.

50. WITH this section commences the work for the eclipse itself. Where figures are to be given, the computation should be made with five-place logarithms to seconds of arc; but where no figures are to be given, four-place logarithms to minutes of arc are sufficient, for these quantities are required only for plotting on the chart.

CHAUVENET'S formulæ for this section are as follows:

$$\left.\begin{aligned} M_0 \sin M_0 &= x_0 \\ M_0 \cos M_0 &= y_0 \end{aligned}\right\} \tag{54}$$

$$\left.\begin{aligned} n \sin N &= x_0' \\ n \cos N &= y_0' \end{aligned}\right\} \tag{55}$$

$$\sin \psi = \frac{m_0 \sin (M_0 - N)}{p + l} \tag{56}$$

$$\tau = \frac{p + l}{n} \cos \psi - \frac{m_0}{n} \cos (M_0 - N) \tag{57}$$

$$T = T_0 + \tau \tag{58}$$

$$\gamma = N + \psi \tag{59}$$

$$\tan \gamma' = \rho_1 \tan \gamma \tag{60}$$

$$p = \frac{\sin \gamma'}{\sin \gamma} = \frac{\rho_1 \cos \gamma'}{\cos \gamma} \tag{61}$$

$$\gamma = N + \psi \tag{62}$$

$$\tan \gamma' = \rho_1 \tan \gamma \tag{63}$$

$$\tan \varphi = \frac{\tan \varphi_1}{\sqrt{1 - e^2}} \tag{64}$$

$$\left.\begin{aligned} \cos \varphi_1 \sin \vartheta &= \sin \gamma' \\ \cos \varphi_1 \cos \vartheta &= - \cos \gamma' \sin d_1 \\ \sin \varphi_1 &= \cos \gamma' \cos d_1 \end{aligned}\right\} \tag{65}$$

$$\omega = \mu_1 - \vartheta \tag{66}$$

Use five-place logarithms and to seconds throughout. Compute $\log m_0$ and M_0 once for the epoch hour, then two approximations are necessary for the times. In the first, take the quantities for the epoch hour and p, which is unknown, equal to unity. There will be but one column for this; but when $\sin \psi$ is reached there will be two angles resulting: one greater than 90° with its cosine negative, and the other less than 90° with its cosine positive. The angles may be either + or —; and the double sign of $\cos \psi$ will give two values

for the times, which may be placed below one another in the same column. Log n is not needed in the numerator, so that log $\frac{1}{n}$ may be gotten instead.

For the second approximation compute in two columns, taking out in each the several quantities for the times, T, already found and compute the quantity, p, for beginning and for ending. Then with x_0' and y_0' compute n and N for these times, and proceed with the formulæ for T, the final times. For beginning, ψ is taken as $> 90°$ with its cosine negative, and for ending $< 90°$ with its cosine positive. The quantity sin ψ for beginning, indeed, gives two values, but the acute value for ending will be less accurate than the first approximation, for the quantities from which it is derived are taken further from the time of ending than in the first approximation. Similar remarks apply to the ending. These other values could be used as a check, but as the final times agree so nearly with the first approximation, this is a partial check in itself.

In the second approximation at least *four* decimals of a minute are required to get the longitude correctly, so that the terms should be carried out to *five* decimals, though only *one* is given in the *Almanac.*

Having the final times, omit Nos. 59, 60, 61, and proceed with 62. γ and γ' differ very slightly, so that the signs of sin γ' and cos γ' are readily known from γ. Sin d_1 is interpolated in the tables and also in our example to every 10 minutes, and should be still further interpolated to the times T; cos d_1 is given for every hour. The first and second of No. 64 both give cos φ, and I always get both values; for if there is any inconsistency in the above work, it will show itself, but this will not detect other errors. And also sin φ_1 and cos φ_1 should give the *same angle* within one unit of the last decimal of logarithms. By *consistency*, I mean that all the sines and cosines used in these equations must belong to the same angle; and the additions of the logarithms must be correct. Log $\frac{1}{\sqrt{1-e^2}}$ is found among the constants (Art. 23). ϑ is the local apparent time of the phenomena throughout this discussion, and can be reduced to mean time by applying the equation of time. μ_1 must be taken from the eclipse tables for the 10^m preceding the time, the proportional parts for the time, T, can be taken from Table VIII. and added to it. These values for beginning and ending may be checked by their difference agreeing with the change of μ_1 for the

difference of the times. The larger term of the difference can be taken from the tables, and the proportional part from Table VIII.

p is the radius of the earth for the place of contact and may be checked by Table IV., taking φ as the argument. The logarithms should agree within one unit, or at most two units, of the last place of decimals. T results in decimals of an hour, which can be reduced to minutes; but the decimals are used as the argument for μ_1, in Table VIII. It is preferable and much easier to take ϑ as minus for beginning and plus for ending, and greater or less than 90°; the angle will then have the same sign as its sine.

51. Generally throughout this discussion angles that are measured from the principal meridian, such as M, γ, ψ in the Maximum, are $< 90°$ in the northern hemisphere—that is, north of the point Z, and $> 90°$ in the southern. ϑ in this respect depends upon the elevation of the pole. And they, with also ϑ, if taken less than 180° and either $+$ or $-$ are always $-$ west of the axis of Y, and always $+$ when east. So, if they are run around from 0° to 360°, the reader will be likely not to see this symmetry. The angle ψ used in this section is, however, quite a different angle from ψ in the Maximum Curve alluded to a few lines above.

52. *Middle of the Eclipse.*—CHAUVENET does not give this for the eclipse generally, but it may be found thus—on page 469, vol. i., of his *Astronomy*, is the following pair of equations (not numbered), which he develops into the formulæ for extreme times:

$$(p+l)\sin(M-N) = m\sin(M_0-N)$$
$$(p+l)\cos(M-N) = m_0\cos(M_0-N) + n\tau$$

If we take the sum of the squares of these we have

$$(p+l)^2 = m_0^2\sin^2(M_0-N) + [m_0\cos(M_0-N) + n\tau]^2 \quad (67)$$

Now as the shadow approaches, m diminishes until at its least value the eclipse is evidently the *greatest*; m then increases and the eclipse becomes less.

In the above equation M_0 and m_0 are taken at the epoch hour; N varies slowly, and is sensibly constant for small intervals of time; the last term is essentially positive, being a square; and $m = p + l$ is, therefore, a minimum when the last term is zero, whence we have

$$\tau = -\frac{m_0}{n}\cos(M_0-N) \quad (68)$$

And for the time,

$$T = T_0 + \tau. \quad (69)$$

This term is already given in the preceding computation, and the mean of the two values for beginning and ending should be taken. The time of greatest eclipse is considered as the *Middle of the Eclipse* as it results from the variable quantities taken at the middle time. It is usually not equidistant between the two extremes, but it is consistent with the principle that the quantities for computing a phenomenon should be taken near the time of that phenomenon. When the path is much inclined, the compression of the earth will be different for the two points of contact, and this will affect the times differently. No figures are given for this time in the *Almanac* unless the eclipse is partial, to which the reader is referred, Art. 78, Section VII., on Maximum Curve.

The computer will find that the Middle, as given by the extreme times of Central eclipse and of Limits of Penumbra, each give slightly different values.

53. *Criterion for Partial Eclipse.*—This is obtained from the computation of extreme times generally.

$$m \sin (M - N) > p. \tag{70}$$

This expression gives the distance of the path from the centre sphere, and is already computed. p is the radius of the earth at this point, which may vary from $\log \rho$ at $90° = 9.9985$ to 9.9987 at $70°$ (Table IV.).

54. *Example of the Computation.*—This is given just as it was made, except that the first approximation is omitted. It is in one column with double values after getting $\sin \psi$, one placed below the other to economize space. $p + l$ is taken as $1 + l$.

The times from the first approximation and the angle ψ, which are needed in the rest of the work, are placed after m_0. Just before getting the final times the figures (1) (2) denote the two terms of equation (57). The proportional parts for μ_1 are given in this example, among them the small term $+ 2''$. The table of proportional parts, which is based on a change of $\mu_1 = 15° \ 0^m \ 0^s$ in one hour, but this varies slightly—sometimes more, sometimes less. In this example (see Tables) it is about $18''$ in one, or $3'$ in ten minutes. The time of beginning is $6^h \ 7^m.8$, and the change in this $7^m.8$, which the proportional parts do not take account of is the $+ 2''$, mentioned above. For ending this correction is zero.

EXTREME TIMES GENERALLY, TOTAL ECLIPSE, 1904, SEPTEMBER 9.

(54) x_0	+ 8.98664	
y_0	− 9.30179	
tan M_1	9.68485	
M_0	+ 154 10 22	
sin	9.63914	
	9.34750	
cos	9.95429	
log m_0	+ 9.34750	
(55–58) from first approximation.		
T	6 7.806	11 20.958
ψ	+ 173 54 30	+ 6 5 30
(59) γ	+ 281 8 26	113 19 26
(60) sin γ	9.99174	9.96297
tan γ	0.70570	0.36536
tan γ'	0.70424	0.36391
(61) sin γ'	9.99168	9.96275
p	+ 9.99994	+ 9.99978
l	+ 9.72631	9.72634
$l - l\,A$	0.27363	0.27344
B	0.45905	0.45893
log $(p+l)$	+ 0.18536	+ 0.18527
(55) x_0'	+ 9.74644	+ 9.74640
y_0'	− .23780	− 9.23810
tan N	.50864	9.50830
N	+ 107 13 24	+ 107 14 10
sin	9.98007	9.98004
log n	9.76637	9.76636
cos	9.47143	9.47175
log 1 : n	+ 0.23363	+ 0.23365
(56) $M - N$	+ 46 56 58	+ 46 56 12
sin ( )	+ 9.86377	+ 9.86368
cos ( )	+ 9.83419	+ 9.83430
m sin	+ 9.21127	+ 9.21118
sin ψ	+ 9.02591	+ 9.02591
ψ	+ 173 54 24	+ 6 5 36
(57) cos ψ	− 9.99754	+ 9.99754
log (1)	− 0.41653	+ 0.41646
(2)	+ 9.41535	+ 9.41545
Nos. (1)	− 2.60933	+ 2.60894
−(2)	− 0.26022	− 0.26029
τ	− 2.86955	+ 2.34865
(58) T	6.13045	11.34865
	$6^h\ 7^m.8270$	$11^h\ 20^m.9190$

(62) γ	+ 281 7 48	+ 113 19 46
(63) tan γ	0.70611	0.36524
tan γ'	0.70465	0.36378
sin γ'	− 9.99170	+ 9.96273
cos γ'	+ 9.28705	− 9.59894
(64) sin d_1	+ 8.96623	+ 8.95963
cos ϕ cos ϑ	− 8.25328	+ 8.55857
tan ϑ	1.73843	1.40416
ϑ	− 91 2 47	+ 87 44 31
sin	9.99992	9.99966
	9.99178	9.96307
cos	8.26151	8.59550
cos ϕ_1	+ 9.99177	+ 9.96307
cos d_1	+ 9.99813	+ 9.99813
sin ϕ_1	+ 9.28518	− 9.59707
tan ϕ_1	+ 9.29341	− 9.63400
(65) tan ϕ	+ 9.29488	− 9.63547
ϕ	+ 11 9 18	− 23 21 50
(66) μ_1 Tables	90 40 49	170 42 23
pp	1 45	0 13 30
	12 0	9
Table VIII.	18	8
	6	0
	+ 2	
μ_1	92 38 15	170 56 10
(66) ω	+ 183 41 2	+ 83 11 39
	− 176 18 58	= east.

Middle.

(69) τ = (2) mean value		− 0.260
(70) T		8.740
		8^h 44.40

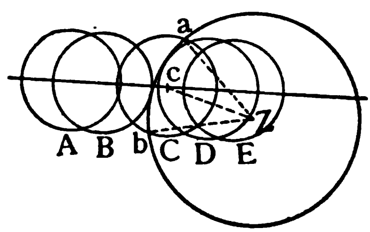
55. *Internal Contacts.*—These are alluded to by CHAUVENET on pages 468 and 470, but are not clearly explained. As these and the

points of beginning and ending are special cases of the Rising and Setting Curve, we will borrow an equation from the next section to explain them.

$$\sin \tfrac{1}{2}\lambda = \pm \sqrt{\frac{(l+m-p)(l-m+p)}{4mp}} \tag{71}$$

In this equation all the quantities are positive; p is the radius of the earth, l that of the shadow, and m, as we will see presently, is the distance between the two centres; p and l are sensibly constant, so that m is the only variable. As the shadow approaches the earth, m is very large, as in the first position, A, in Fig. 5 annexed; hence, the second factor of this equation is negative, and the value of λ imaginary. It is not until this factor becomes o, position B, that λ has a real value, then $=0$, which is the beginning or end of an eclipse, for which the condition is

FIG. 5.



$$m = p + l. \tag{72}$$

As the shadow advances it crosses the primitive circle in the points a and b, and λ is the angle which these lines aZ and bZ make with the line cZ, position C. If internal contacts exist, the shadow will become tangent to the primitive circle on the *inside*, position D, and the first factor will become 0. The condition for these is then

$$m = p - l. \tag{73}$$

As the shadow advances further, m becomes smaller, and $l + m$ is smaller than p numerically, and this factor then becomes negative and λ again imaginary, position E; and remains so until m, having passed the centre of the sphere, begins to increase, when $l + m = + p$, and we have the second internal contact, which again gives equation (73), and the above succession of phenomena is reversed. It is a beautiful equation.

The formulæ for the internal contacts differ from those for external only in the substitution of $p - l$ for $p + l$ in equations (56) and (57). They are not computed for the *Nautical Almanac* nor shown on the charts; we will, however, recur to them again before closing this section.

The internal contacts exist in the eclipse of 1904, Sept. 9, occur-

ring at about $7^h\ 58^m$ and $9^h\ 29^m$, as shown by scale in Fig. 2, Plate I. They do not exist in the eclipse of 1860, July 19 (Fig. 3, Plate II.).

56. The reader must not confuse these four contacts with the four contacts mentioned by CHAUVENET, vol. i., p. 440, Art. 288, which are the contacts of the umbral and penumbral cones with the earth, and near the points K and L (Figs. 2 and 3). The first and last are the same as we have just considered.

57. *Check on the Extreme Times.*—A rigorous check upon the extreme times is furnished by the equation (72), computing m for the beginning and ending and comparing with $p + l$.

Place

$$\left.\begin{array}{l} m \sin M = x \\ m \cos M = y \end{array}\right\} \qquad (74)$$

Then check

$$M = \gamma = N + \psi \qquad (75)$$

$$\log m = \log (p + l) \qquad (76)$$

Compute with five-place logarithms x and y for each of the times T, using *four* decimals of a minute; then compare M with γ, which is used for the latitudes and longitudes in the computation above; and compare $\log m$ with $\log (p + l)$, which is also there given. The former should agree within 3″ or 4″, and the latter within 1 or 2 units of the last decimal place. If the checks agree, it shows that these quantities and also the times are correct.

In the example following, the factor of T is a fraction of ten minutes.

CHECK ON EXTREME TIMES GENERALLY.

	Beginning.		Ending.	
	Nos.	log.	Nos.	log.
(74) T, factor,	6^h 7.8270	9.89360	11^h 20.9190	8.96332
Δx	+ 0.09259	8.96825	+ 0.09292	8.96811
	+ 0.07275	8.86185	+ .00854	7.93143
x tables	− 1.57629		+ 1.39826	
x	− 1.50354	− 0.17712	+ 1.40680	+ 0.14823
Δy	− 0.02880	8.45939	− 0.02884	8.46000
	− 0.02254	8.35299	− 0.00265	7.42332
y table	+ 0.31835		− 0.60406	
y	+ 0.29581	+ 9.47101	− 0.60671	− 9.78298
tan M				
(75) M	281 7 48	0.70611	113 19 45	0.36525
check γ	48		46	
sin N		9.99176		9.96296
log m		0.18536		0.18527
(76) Check log $p + l$		0.18536		0.18527

The extreme times may also be found in a very simple manner from equation 72. Place it under the following form:

$$\Delta = m - (p + l).$$

Here p is constant and l nearly so, and m the only variable; the extreme times occur when $\Delta = 0$. We can get p, the radius of the earth, sufficiently exact from Fig. 2, the angle RZA by protractor measures 11°, which is nearly the latitude of the place, since the pole is but slightly elevated. Table IV. gives the earth's radius, which in numbers is 0.99988; l from the eclipse tables is 0.53248, and $p + l$ is 1.53236. Next find x and y for every ten minutes; the differences will have to be extended back from 6^h for the first date. With these, using five-place logarithms, compute M and $\log m$ and get m in numbers.

Tabulating these, we have as follows:

Time.	m.	$m-(p+l)$.			M.		
5 50	1.70498	+ 0.17262			281° 44′ 54″		
			− 9686			− 19′ 49″	
6 0	1.60812	+ 0.07576		+ 7	281 25 5		− 2′ 33″
			− 9679			− 22 22	
6 10	1.51133	− 0.02103		11	281 2 43		− 3 4
			− 9668			− 25 26	
6 20	1.41465	− 0.11771			280 37 17		

By formula 11, Art. 21, find when this is zero, using five-place logarithms. In numbers the terms are

$$t = \frac{-.075760}{-.096790 - .000045 + .000026} = \frac{.07576}{.09681} = 0.78265,$$

which is a fraction of ten minutes.

Hence, the time is $6^h\ 7^m.8265$,
And from the regular computation we get $6\ \ 7\ .8270$,

which is sufficiently exact to give the longitude correctly. With this time, interpolating M, it is found to equal γ exactly; hence, the latitudes and longitudes can also be found.

58. *Geometrical Illustration.*—We already have the sphere and path of the eclipse projected (Fig. 2, Plate I., and Fig. 3, Plate II.). With a pair of dividers opened to the radius of the penumbra l, place one leg on the path and move it until the other leg sweeps a circle tangent to the sphere at R and S. The point a on the path then gives the time of beginning generally, and similarly b that of ending, and R and S are the points on the earth of first and last contact. The internal contacts, which exist only in Fig. 2, are found in a similar manner with contacts on the inner side of the primitive

circle; they are marked *c d* on the path. The path may now be divided into 10^m spaces, or closer if the scale of the drawing admits, and the times read off; the points of contact, R and S, on the earth's surface, are those given in the *Almanac* for *First* and *Last Contact;* their latitudes and longitudes can easily be gotten from these figures. If we mark 9^h (Fig. 2) and 2^h (Fig. 3) as the epoch hour T_0, and take the coördinates of T_0, the epoch hour, as x_0 and y_0, we have (formula 54) $YZT_0 = M_0$, measured always from the axis of Y toward the right as positive $= + 154° \ 10'$ in Fig. 2, and $- 7° \ 46'$ in Fig. 3, and $T_0Z = m_0$. Likewise the hourly motions give the angle which the path makes with the axis of Y. This is seen from the triangles having H at the right angle, but as y is negative in both figures, the angle in each case is greater than $90°$; and if J be the point where the path crosses the axis of Y, $CJb = N$, and n is the hypothenuse of those right-angled triangles (marked H at the right angle)—the motion of the shadow along the path in one hour. N is also measured from the axis of Y toward the right as positive, and is never negative.

In Plates I. and II., as well as in subsequent ones, the geometrical quantities used in the formulæ, lines, and angles, etc., are marked in the figures with thin skeleton letters, which should not be mistaken for the heavier reference letters. To have the lines and angles marked will generally render the description clearer.

The reader should follow the formulæ now as closely as in the example. $M - N$ is the angle between the path and T_0Z, measured from the path toward the right as positive. Draw ZI a perpendicular to the path, forming T_0IZ, a right-angled triangle in which the angle at $T_0 = (M - N)$, and $T_0Z = m_0$; hence $ZI = m_0 \sin (M - N)$; ZR and ZS are the earth's radii $= p$; aR and bS are the radii of the penumbral shadow l. Hence we have two other right-angled triangles, aZI and bZI, in which aZ and $bZ = p + l$, and $ZI = m_0 \sin (M - N)$ in each, from which we have the angles at a and b given by their sine:

$$\sin \psi = \frac{IZ}{aZ} = \frac{m_0 \sin (M - N)}{p + l} = \frac{IZ}{bZ}.$$

And here we see the ambiguity as to which angle to take—the acute or obtuse value. I have here placed these angles equal to one another, which is sufficiently accurate in the drawing and, moreover, renders the explanation more easy. Then we have

$$(p + l) \cos \psi = aI = Ib.$$

This is reduced from space to time by dividing it by n, the motion of the shadow in one hour along the path. We also have $T_0I = m_0 \cos(M - N)$, and this is also reduced to time by dividing by n. Now we know the time the shadow takes in passing from a *to* I and from I *to* b, also from T_0 to I. Hence the time numerically, and $\tau_0 = -aI - T_0I = +Ib - T_0I$. We know the time at T_0, the epoch hour, so we have the times, $T = T_0 + \tau$.

59. For the middle time, it is readily seen why it is given by the second term of Equation 57.

$$-\frac{m_0}{n}\cos(M_0 - N) = T_0I,$$

which reduces the time from T_0 to the point I, the greatest eclipse, which is also the Middle.

60. The latitudes and longitudes of the points of beginning and ending are next required. The formula 65 can be shown geometrically; but there is a simpler method which can be used here, since all the points lie in the fundamental plane. It will be noticed, however, that $\gamma = N + \psi = CZR$ for beginning, which is always negative if taken less than 180°; and CZS for ending, which is always positive; γ throughout the eclipse formulæ is measured from the northern part of the axis of Y toward the right as positive.

The method of getting the latitude and longitude will be illustrated by only one point—that for ending, which is located in Fig. 2 by the letter S. Revolve the whole sphere 90° around the line CD as an axis; the pole P will fall at P', as in Art. 39. $P'Z$ produced is the earth's axis. S will fall at m in the line CD on the lower surface of the sphere, and E at E'. As the pole is in the principal plane, all the circles of latitude will project in right lines perpendicular to the axis $P'Z$, hence $E'E''$ will be the equator, and mn, crossing the axis at t, and at right angles to it, will be the parallel of latitude of the point S, and the latitude can be measured by a protractor, the arc $E''n$.

For the longitude, resume the sphere in its normal position and revolve it around the line AB until the pole falls in the point Z; the parallels of latitude will then project in circles. We have the radius of the parallel of the point S, which is tn; with this describe an arc, o, from Z as a centre, and it will be the required parallel. During this revolution the point S will revolve in the vertical circle, Sq, below the primitive plane, and fall at s, where

this line meets the circle of latitude. In the present case the line Sq meets the circle at such an acute angle that the point s is rather indefinite; and another construction is necessary to determine how far S has revolved. With the sphere in its normal position, intersect it by a plane through Sq, which will cut a small circle from the sphere, whose diameter is Sq. This circle revolve down so that the part above the line AB falls at z. We can now see the revolution. The pole was revolved through an arc $90° - d = 90° - 5° \ 15' = 84° \ 45'$. S at the same time revolved through the same arc and fell at r. Its motion along the line Sq is the sine of this angle found by projecting r on the line at s. Therefore, we have the hour angle, DZs, measured in a positive direction. The Greenwich Meridian at the time of ending is on the circle of the sphere about one-third distant from 11^h to 12^h, and μ_1 is the angle from this point to D; and the longitude, the arc from the position of Greenwich to the line ZS extended.

The drawing is too crowded for more lines; but the results for the inner contacts, taken from the figure by scale and protractor, are as follows:

T	2d Contact	$7^h \ 58^m.5$	3d Contact	$9^h \ 28^m.7$
φ		$-3° \ 30$		$-32° \ 30$
ϑ		$-89 \ 30$		$+85 \ 30$
μ_1		$+120 \ 20$		$142 \ 50$
ω		$\begin{cases} +209 \ 50 \\ -150 \ 10 \end{cases}$		$+57 \ 20$

They plot well on the chart in the *Nautical Almanac*, which is reproduced in Fig. 17, Plate VI.

Comparison of the Times of the Contact.			Differences with the Middle.
	First Contact	$6^h \ 7^m.8$	1 36.6
	Second Contact	7 58 .5	0 45.9
	Middle	8 44 .4	
	Third Contact	9 28 .7	0 44.3
	Last Contact	11 20 .9	1 36.5

SECTION VI.

RISING AND SETTING CURVE.

61. CHAUVENET's formulæ are as follows:

$$m \sin M = x$$
$$m \cos M = y \tag{77}$$
$$\sin \tfrac{1}{2}\lambda = \pm \sqrt{\frac{(l+m-p)(l-m+p)}{4pm}} \tag{78}$$
$$\gamma = M \pm \lambda \tag{79}$$
$$\tan \gamma' = \rho_1 \tan \gamma \tag{80}$$
$$p = \frac{\sin \gamma'}{\sin \gamma} = \frac{\rho \cos \gamma'}{\cos \gamma} \tag{81}$$
$$\cos \varphi_1 \sin \vartheta = \sin \gamma'$$
$$\cos \varphi_1 \cos \vartheta = - \cos \gamma' \sin d_1 \tag{82}$$
$$\sin \varphi_1 = \cos \gamma' \cos d_1$$
$$\tan \varphi = \frac{\tan \varphi_1}{\sqrt{1-e^2}} \tag{83}$$
$$\omega = \mu_1 - \vartheta \tag{84}$$
$$\left.\begin{array}{l} m \sin (M-E) < p \sin (\gamma - E), \text{ Eclipse beginning,} \\ m \sin (M-E) > p \sin (\gamma - E), \text{ Eclipse ending.} \end{array}\right\} \tag{85}$$
$$\left.\begin{array}{l} \vartheta \text{ between } 0 \text{ and } -180°, \text{ The sun rising,} \\ \vartheta \text{ between } 0 \text{ and } +180°, \text{ The sun setting.} \end{array}\right\} \tag{86}$$

CHAUVENET intends these formulæ to be used rigorously with five-place logarithms, to seconds of arc, making two approximations; taking in the first $p =$ unity, and repeating the work for λ and γ; then proceeding to the latitudes and longitudes.

62. There is, however, no necessity for this exactness, as the curve is generally only needed for plotting on a chart, and the following simplifications may be made: Neglect the compression of the earth, then $e = 0$, and the following changes result: $p = 1$, $\rho_1 = 1$, $\gamma' = \gamma$, $\varphi_1 = \varphi$, and take d instead of d_1. This will give results almost or quite as close, as can be seen on a projection of the sphere of sixteen inches radius, or thirty-two inches in diameter. Moreover,

equation (85) need never be computed, since both (85) and (86) are known when the curves are plotted on a chart. And as d never varies more than 3′ or 4′ of arc, it may be taken as constant, using the value at the epoch hour.

For these approximations, four-place logarithms to minutes will be sufficient, and compute for every ten minutes between the extreme times. The double sign of λ will give two points to the curve; the computation can be made, using one value throughout all the 10-minute columns, finding φ and ω; then computing with the other value. (For the derivation of this equation (78), see Art. 89.) No example is given of the use of these formulæ, as they are so similar to those of the preceding section.

If a pair of dividers be opened to the radius l and one leg moved upon the path to the successive 10-minute points, the other leg will generally be near enough to touch the primitive circle from each of these 10-minute points. These points on the circle are the two positions given by the formulæ for each of the assumed times on the path. In Fig. 3, Plate II., the curve is seen to be continuous, the centre line being distant from the primitive circle at no part of its path as much as l; but in Fig. 2, Plate I., the path lies so near the centre of the earth that it is at some of its parts at a greater distance than l from the primitive circle; consequently the dividers cannot reach this circle, and there is no rising and setting curve to these times of the path. By moving the dividers successively over the 10-minute points and ascertaining which point on the primitive circle the other leg reaches, the reader will see how the inner constants are formed in Fig. 2, and the reason also why they do not exist in Fig. 3.

63. The above formulæ may be partially illustrated as follows: In Fig. 6—

The coördinates of centre of shadow are $m \sin M = x$

$m \cos M = y$

Coördinates of point of the curve on the earth, $p \sin \gamma = \xi$

$p \cos \gamma = \eta$

Fundamental equations of eclipses (CHAUVENET, i., 447, 449), $l \sin Q = x - \xi$

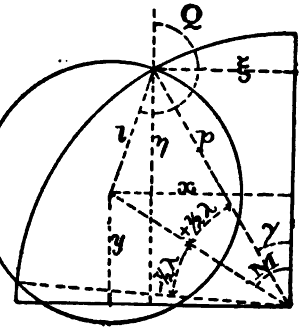
$l \cos Q = y - \eta$

Δ is here equal to l.

Q is the angle of position of the centre of the shadow from any point on the cone of shadow, and is measured from the axis of Y toward the right as positive. The meanings of the quantities are

clear from the figure, and the transformations to equation (78) for $\frac{1}{2}\lambda$ is merely to get this angle, a convenient form for use. (See Art. 64 following.)

FIG. 6.



As the radius of the shadow is but little more than half the earth's radius, λ can never much exceed 30° or 35°.

64. *Dr. Hill's Formulæ.*—The following formulæ were devised by Dr. George W. Hill for this curve. Their derivation is given in Section VIII., as they result directly from formulæ for the outline curves:

$$\left.\begin{aligned} m \sin M &= x \\ m \cos M &= y \end{aligned}\right\} \tag{87}$$

$$\cos(\theta - M) = \frac{m^2 + 1 - l^2}{2m} \tag{88}$$

$$\left.\begin{aligned} \cos\varphi \sin\vartheta &= \sin\theta \\ \cos\varphi \cos\vartheta &= -\sin d \cos\theta \\ \sin\varphi &= \cos d \cos\theta \end{aligned}\right\} \tag{89}$$

$$\omega = \mu_1 - \vartheta \tag{90}$$

Use four-place logarithms to minutes, and compute for every ten minutes between the extreme times. As x and y are required to five places of logarithms in the central curve, it is well to get them here to five places for the times between the limits when the centre of the shadow crosses the primitive circle; then tan M need be taken only to four decimals. $1 - l^2$ is a constant for the curve; $(\theta - M)$ has two values, + and —, as in CHAUVENET'S formula, which gives two points. As d varies only a few minutes during the whole eclipse, sin d and cos d may be taken as constant near the middle time. Finally, instead of getting tan φ, take the angle from the sine and notice whether the cosine gives the same angle. Their agreement will check several errors, and the computer may remember that two points separated by 10′ or 15′ of arc can hardly be seen on a chart of which the radius of the sphere is sixteen inches. The heavy lines on the eclipse drawings prepared for the *Almanac* are about 15′ wide.

65. *Example.*—One point of the eclipse of 1904, Sept. 9, at $7^h\,10^m$, is taken, giving two geographical positions. Each column should be

headed with the hour and 10^m. The example here given should be carried out in one column. The constants, which are used on a slip of paper, are here written in the margin of the work. Repetition of figures throughout the whole eclipse computation is avoided as much as possible. With $\log \frac{1}{2} = 9.6990$, $\log \dfrac{1-l^2}{2}$ is gotten as a constant and added to the quantity B of the addition and subtraction logarithms. $(\theta - M)$ has the double sign, which gives two values to θ and thence two geographical positions for the two branches of the curve at this time.

Example, Rising and Setting Curve.

			$7^h\ 10^m$.	
(87) Numbers	$-$ 0.9256	$\log x$	$-$ 9.9664	
	$+$ 0.1167	$\log y$	$+$ 9.0671	
		$\tan M$	0.8993	
		M	$-$ 82 49	
		$\sin M$	9.9966	
		$\log m$	$+$ 9.9698	
(88)		$\log m^2$	$+$ 9.9396	
$\log (1-l^2)$	$=$ 9.8552	$l - l$	A 0.0844	
			B 0.3453	
$\log \dfrac{(1-l^2)}{2}$	9.5542	log	$+$ 9.8995	
		$\cos (\theta - M)$	9.9297	
		$(\theta - M)$	$\mp$ 31° 44′	
(89)		θ	$-$ 114 33	$-$ 51 5
		sin	$-$ 9.9588	$-$ 9.8910
		cos	$-$ 9.6186	$+$ 9.7981
$\log \sin d$	$+$ 8.9612	$\cos \phi \cos \vartheta$	$+$ 8.5798	$-$ 8.7593
		$\tan \vartheta$	1.3790	1.1317
		ϑ	$-$ 87 36	$-$ 94 14
		sin	9.9996	9.9988
		$\cos \phi$	$+$ 9.9592	$+$ 9.8922
$\log \cos d$	9.9982	$\sin \phi$	$-$ 9.6168	$+$ 9.7963
		ϕ	$-$ 24° 27′	$+$ 38° 44′
(90) μ_1	108° 11′	ω	$+$ 195 47	$+$ 202 25

66. *The Node or Multiple Point.*—This is the point where the two branches of the rising and setting curve cross one another. It is seen in Fig. 3, but not in Fig. 2, as it generally does not exist when the inner contacts are formed. The points of the two curves are passed over by the cone of shadow at widely different times, and the method of its formation is given in the next section in connection with the Maximum Curve. It is an unimportant point, but should be properly located on the chart. It is very near the meridian on

which the sun is central at local apparent noon, so we may take with sufficient exactness,

$$\varphi = 90^\circ - d, \tag{91}$$

$$\omega = \mu_1. \tag{92}$$

If the shadow goes over the pole instead of the latter,

$$\omega = \mu_1 + 180^\circ. \tag{93}$$

67. *Singular Forms of the Rising and Setting and Other Curves.*— The two most usual forms of the rising and setting curve are shown in Fig. 3, the distorted figure 8, and in Fig. 2 the two separate branches connected by two limiting curves, which are more readily recognized in Fig. 17, Plate VI., and Fig. 18, Plate VII., reproduced from the eclipse charts. Other special forms are as follows:

A. When the rising and setting curve breaks to form two limiting curves, the break does not necessarily take place at the node. This is well illustrated in the eclipse of 1894, Sept. 28 (Fig. 7, Plate IV.), at *A*, where the whole shadow very nearly fell upon the earth; the rising and setting curve contracted at this point, then diverged before crossing at the node. If the shadow had fallen a very little further north, the break would have occurred at the point *a*, forming *three* branches, one of which would extend from some point near *a* to the node. The form of the curve is shown in the figure.

B, partial eclipse, March 26, 1884. Here the southern limit of shadow passed but a short distance to the right of the axis *Y*, consequently that branch was very small, as shown at *B*, Fig. 7. The next eclipse of the series, 1902, April 8, for which no chart was given in the *Almanac*, had but one branch of the rising and setting curve shown at *b*.

C. Let this line (Fig. 7) represent the centre line of the annular eclipse of 1891, June 6; the pole *p* being below the line, the shadow has consequently passed over and beyond the pole, and the "Central Eclipse at Noon," as given in the *Nautical Almanac*, is more properly "Central Eclipse of the Midnight Sun." In this eclipse also another peculiarity is noticed, the path of the Annulus passed from *East* to *West*.

Let *D* (Fig. 7) represent the centre line of the annular eclipse, 1896, Feb. 13. It is seen that it is so far south that it does not cross the principal meridian; there is consequently no "Eclipse at Noon," as generally given in the *Almanac*.

E. This line (Fig. 7) represents the path of the Total-Annular Eclipse, 1890, Dec. 11. Here the vertex of the umbral cone fell

just above the fundamental plane, so that the ends of the eclipse were annular and the middle total, the limiting curves crossing, as shown in the figure. Professor Newcomb selected the name Central for this, as given in the *Almanac.*

F (Fig. 7). The central path of the annular eclipse, 1874, Oct. 7, passed over so small a portion of the earth that the line of central eclipse lay wholly within the rising and setting curve, as shown at *F.*

G. The partial eclipse of 1862, Nov. 21, presents a curious anomaly: the southern pole of the earth, p', was elevated almost to its extreme limit, the shadow, as much as fell on the earth, was wholly beyond the pole, p', so that the *northern* limiting curve was the nearest point of this eclipse to the *south* pole, and all of the shadow passed over the earth from east to west. No chart is given in the *Almanac,* but sufficient latitudes and longitudes to construct the eclipse as here shown.

H. The limiting lines of the central eclipse sometimes show a peculiarity; the angle E at the ends skews the points of this curve so much at the ends that the southern point, h, may have a *greater* latitude than the corresponding northern point, as the case of the total eclipse of 1896, Aug. 8, which is the same eclipse as shown in the chart (Plate VII., Fig. 18). See also Art. 139.

SECTION VII.

MAXIMUM CURVE.

68. Within the space formed by the two branches of the rising and setting curve, on which the eclipse commences and ends in the horizon, there must be a succession of places where the eclipse is seen at its maximum, which forms the curve known as the *Maximum of the Eclipse in the Horizon.* Chauvenet's formulæ are as follows:

$$\left.\begin{aligned} m \sin M &= \eta \\ m \cos M &= y \end{aligned}\right\} \tag{94}$$

$$\sin \psi = \frac{m \sin (M - E)}{p} \tag{95}$$

Take here

$$\log p = \tfrac{1}{2} \log \rho_1 \tag{96}$$

Δ to be positive,

$$\pm\Delta = m\cos(M-E) - p\cos\psi \tag{97}$$

$$\Delta < l \tag{98}$$

$$\gamma = \psi + E \tag{99}$$

$$\tan\gamma' = \rho_1 \tan\gamma \tag{100}$$

$$p = \frac{\sin\gamma'}{\sin\gamma} = \frac{\rho_1\cos\gamma'}{\cos\gamma} \tag{101}$$

Then repeat from Equation (95) for a correct value of γ'

$$\left.\begin{aligned} \cos\varphi_1\sin\vartheta &= \sin\gamma' \\ \sin\varphi_1\cos\vartheta &= -\cos\gamma'\sin d_1 \\ \sin\varphi_1 &= \cos\gamma'\cos d_1 \end{aligned}\right\} \tag{102}$$

$$\tan\varphi = \frac{\tan\varphi_1}{\sqrt{1-e^2}} \tag{103}$$

$$\omega = \mu_1 - \vartheta \tag{104}$$

As in the preceding section, there is no need of such rigor in these equations; CHAUVENET'S method is, however, given in full. Two approximations are necessary, getting p correctly from the first; then repeating equations (95), (99), (100), and (101), and the remainder for the latitudes and longitudes. Four-place logarithms are sufficient.

69. *Remarks on Formulæ and Chauvenet's Text.*—Some of these formulæ may need explanation to the beginner, and one paragraph of CHAUVENET'S text needs revision, as it gives a wrong explanation of the beginning and ending of this curve.

The equation $\log p = \frac{1}{2}\log\rho_1$ is a mean value derived thus: On page 477 of CHAUVENET we have these equations:

$$\left.\begin{aligned} p\sin\gamma &= \xi & \qquad \eta_1 &= \frac{\eta}{\rho_1} \\ p\cos\gamma &= \eta = \eta_1\rho_1 & \qquad \xi^2 + \eta_1^2 &= 1 \end{aligned}\right\} \tag{105}$$

Adding the squares

$$^2\sin^2\gamma + \frac{p^2}{\rho_1^2}\cos^2\gamma = \xi^2 + \eta^2 = 1$$

The extreme values of γ are 0 and $90°$.

$$\text{If } \gamma = 0 \qquad p = \rho_1 \qquad\qquad \text{If } \gamma = 90° \qquad p = 1$$

The mean of these is

$$p = \frac{\rho_1 + 1}{2}, \text{ whence} \qquad p^2 = \frac{\rho_1^2 + 2\rho_1 + 1}{4} = \rho_1$$

In the latter equation ρ_1 is larger than its square and smaller than unity, therefore the numerator is approximately $4\rho_1$.

Passing to logarithms

$$2 \log p = \log \rho_1, \text{ whence} \qquad \log p = \tfrac{1}{2} \log \rho_1.$$

On page 476 of CHAUVENET, lines 10 and 11, there is an error of signs—those between the terms should be transposed: In the first equation the sign should be $+$, and in the second, $-$.

In Art. 309, page 478, is the following paragraph: "The limiting times between which the solution is possible will be known from the computation of the rising and setting limits, in which we have already employed the quantity $m \sin (M - E)$; and the present curve will be computed only for those times for which $m \sin (M - E) < l$. These limiting times are also the same as those for the northern and southern limiting curves, which will be determined in Art. 313."

The paragraph is wrong throughout. The limits of the rising and setting are really the *extreme times;* but by these words CHAUVENET means the extreme times of the northern and southern limiting curves; and these are *not* the limiting times of the maximum in all cases. They are in CHAUVENET'S example, but not in the eclipse of 1904, September 9. The equation $m \sin (M-E) < l$ should read $<$ unity. Compare page 479, line 2.

70. *The Maximum Curve* is correctly explained thus: As the earth is moving under the shadow in some different direction, the projection of the path over the earth's surface will not be its projection on the fundamental plane, but will be, perhaps, the resultant of the two motions. The northern and southern limiting curves will consequently not be formed by the element of the cone normal to the path, but by some other element swung around from the right angle by the angle $E +$ some variable quantity. At the beginning

and ending of these curves this variable is zero, and $Q = E$ or $Q = E \pm 180°$ (CHAUVENET, i., p. 481, Art. 311). Hence, for the beginning of the Maximum Curve, conceive a line crossing the path of the eclipse, making the variable angle E with the axis Y. As this line moves along the centre line, its intersection with the earth's sphere will generate the Maximum Curve in all cases; and there are four special positions to be examined.

In Fig. 8, Plate V., which is a reproduction of Fig. 2, the eclipse of 1904, September 9, draw the line ZW, making the angle $E = CZW$ with the axis. Also draw ab from the path tangent to the sphere at b and parallel to ZW, also making the angle E with the axis. Then in Equations (94) and (95) we have, angles measured from the axis toward the *left* being negative:

$CZa = M$, which is negative in this case.
$CZW = E$, which is positive.
$WZa = M - E$, which is negative and obtuse.
$Za = m$.
$ac = m \sin (M - E)$, negative,
$= Zb$, since ab is parallel to cW
$= \rho$, the radius of the earth's sphere.

Hence $\sin \psi = \dfrac{m \sin (M - E)}{p} = \dfrac{\rho}{p} = -1$, taken negatively from Z for beginning.
$= WZb = 90°$ in this position.

Here we see the errata made in CHAUVENET'S clause above quoted; as $p = \rho$, the earth's radius, $m \sin (M - E)$ equals *unity*, very nearly.

Just before the line ab became tangent to the sphere, m had a larger value, $\sin \psi >$ unity, and ψ therefore imaginary, giving no points to the curve. This is the first position above referred to. The maximum curve therefore begins at b when $\sin \psi =$ unity. Immediately after the first contact, as the line ab advances, it cuts the sphere in two places, and there are *double points* to the maximum curve, which CHAUVENET refers to, but does not clearly explain. These double points exist until the line ab takes the position de, in this example, in which e is the point of beginning of the northern limiting curve, which CHAUVENET vaguely calls the "limiting times" of the maximum curve. A double point at $7^h\ 10^m$ is given in the example below.

71. To digress for a moment, it will be noticed that these points, marked VW in Fig. 3, are the extreme points of the southern limiting curve; and in this figure are the "limiting times" of the maximum, as CHAUVENET writes while having this eclipse in his mind, but the statement is not general as applying to all eclipses.

72. Take a second position, the line de through the extreme point e of the northern limiting curve. The angles M and E are hardly changed, since the former position

$$\begin{aligned} WZd &= M - E, \text{ negative and obtuse.} \\ Zd &= m. \\ dc &= m \sin (M - E), \text{ negative,} \\ &= d'Z. \\ \sin \psi &= \frac{d'Z}{Ze} = \frac{d'Z}{p}. \end{aligned}$$

This equation gives two values of ψ, according as the cosine is taken positive or negative, and these give the double points to the curve when they exist. We can take the two cases together; laying off the arc $bl = be$, we have

$$\begin{aligned} &\text{With } \cos \psi +, \ \psi = WZe, \\ &\text{With } \cos \psi -, \ \psi = WZl. \end{aligned}$$

And by formulæ (97) and (98) for the criterion,

$$\begin{aligned} Zd &= m, \\ Zc &= m \cos (M - E) \text{ and negative.} \\ &\left\{ \begin{aligned} Zf &= p \cos \psi, \text{ positive, since } \cos \psi \text{ is positive,} \\ Zk &= p \cos \psi, \text{ negative, since } \cos \psi \text{ is negative.} \end{aligned} \right. \\ \varDelta &= m \cos (M - E) - p \cos \psi. \\ &\left\{ \begin{aligned} &= Zc + Zf = cf, \text{ the numerical sum,} \\ &= Zc - Zk = ck, \text{ the numerical difference.} \end{aligned} \right. \end{aligned}$$

In the first case, $\varDelta = l$, as it should, since we have taken the point at which the maximum ends.

In the second case, $\varDelta < l$, and there is a double point to the curve located at l.

As the shadow moves, the northern branch gives $\varDelta > l$, and the condition is not fulfilled; but there are single points to the southern branch, and this is the third position referred to. Single points exist until the line ab reaches the position gh, after which for both branches

$$\varDelta > l,$$

and the condition is not fulfilled by either, yet the formula gives *real points*, though they lie outside of the limits of the curve. In fact, they extend round the whole circle of the earth's sphere. This is the fourth position, and to exclude these values the condition (98) is imposed.

It is sometimes an advantage to compute one of these points outside of the eclipse; or when the points of the maximum curve lie far apart, and if there is no point near the end, a point outside may be utilized to give the direction of the line on the chart, or to check the extreme point.

Generally the criterion (97), (98), need not be computed, and never except toward the ends of the curve. If a plot of the eclipse similar to those here given is made, it will answer every purpose. A point outside of the eclipse may perhaps be inadvertently computed, but an examination of the times on the chart will show if it belongs there. The lines from the path *de*, *gh*, etc., must each make the correct angle of the variable E at the times marked by d and g on the path.

73. Fig. 3, which is CHAUVENET'S eclipse, shows a very simple maximum curve; the lines from the path making the angle E are drawn, which gives the times of beginning and ending. In this eclipse they are the same as the extreme times of the southern limiting curve.

74. For the geographical positions, we have above for the point e the angle ψ.

$$Zed = \psi = eZW, \text{ negative angle.}$$
$$\text{Then } CZe = eZW - CZW = \psi - E = \gamma, \text{ negative angle.}$$

From this equation γ is found for use in getting the geographical positions, which are completely known when this angle is determined and the time known. The formulæ are similar to those in previous sections.

75. *Approximate Formulæ.*—This curve should be placed below the rising and setting curve. Since m and M are already there computed, we then proceed—

$$\sin \psi = m \sin (M - E). \tag{106}$$
$$\gamma = \psi + E. \tag{107}$$
$$\left.\begin{aligned} \cos \varphi \sin \vartheta &= \sin \gamma, \\ \cos \varphi \cos \vartheta &= -\cos \gamma \sin d, \\ \sin \varphi &= \cos \gamma \cos d. \end{aligned}\right\} \tag{108}$$
$$\omega = \mu_1 - \vartheta. \tag{109}$$

As in the former curve, neglecting the compression of the earth, $e = 0_1$, and there results $p = 1$, $\gamma = \gamma'$, $\varphi = \varphi_1$, and $d = d_1$. M and m are to be taken from the rising and setting curve. The criterion (97) and (98) will seldom be needed; it is here given in the example merely to show the double point. γ is always $< 90°$ north of the point Z, and $> 90°$ south of it. If taken less than $180°$, it is negative on the left of the axis and positive on the right. γ and ψ will usually lie in the same quadrant. ψ here has no affinity to the angle ψ in beginning and ending generally.

76. Example—Maximum Curve.

Total Eclipse 1901, September 9.

		7^h 10^m				
$E = +15° 2'$	$M-E$	-97 51	(107)	ψ	$-67°$ 32′	$-112°$ 28′
(106)	sin ( )	-9.9959	(108)	γ	-52 30	-97 26
	cos ( )	-9.1354		sin	-9.8995	-9.9963
	sin ψ	-9.9657		cos	$+9.7844$	-9.1118
	(1) m cos ( )	-9.1052	log sin d + 8.9612 } cos ϕ sin ϑ		-8.7456	$+8.0730$
	(2) cos ψ	∓ 9.5822		tan ϑ	-1.1539	-1.9233
(97) l 0.5326	Nos. (1)	-0.127		ϑ	-94 1	-89 19
	Nos. (2)	∓ 0.383		sin cos	9.9989	0.0000
(98) Double point shown				cos φ	$+9.9006$	$+9.9963$
Since (1) + (2) $< l$			log cos d 9.9982	sin ϕ	$+9.7826$	-9.1100
And (1) − (2) $< l$			μ_1 108° 11′	ϕ	$+37$ 19	-7 24
				ω	$+202$ 12	$+197$ 30

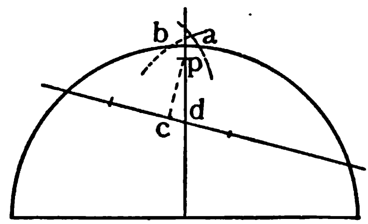
In the criterion the sign of $m \cos (M - E)$ is determined term (1), that of ψ undetermined, except by the criterion, in which, as these terms are to be subtracted, they will generally have the *same sign*, so as to make their difference small and less than l. This will determine the sign of ψ generally in practice. In this example the second value of ψ, that in the last column, is the value corresponding to the succession of points, and the acute value the special case. The criterion determines whether the angle is acute or obtuse, but the sign in either case is determined by that of $\sin \psi$ above in the formulæ.

77. *The Node and Maximum Curve.*—The Maximum Curve does not pass through the node, though in all the earlier charts of the *Almanac* it is drawn so.

The node is formed in the following manner: As the shadow advances, the pole, P, being elevated as in the examples above given,

the preceding arc of the shadow will pass over some point, a, on the *right* of the axis. When the following arc of the shadow approaches, the point a has reached b just as the arc passes over it. The point a in this case revolves *below* the sphere, since the pole is elevated. The motions of the shadow and point a may be considered as constant; and when the centre of the shadow has moved half of its distance and is at c, the point a has also revolved half of its distance and is *on* the meridian below the sphere, the maximum curve is being formed in the fundamental plane, and it is thus seen that it cannot pass through this nodal point. When the centre of the shadow is at d, the point a has moved to the left of the meridian. This is now the meridian of the sun at noon, and the point a is about 180° from it.

FIG. 9.



If the pole is depressed, the shortest arc for the point a to move in will be from b to a, passing *above* the fundamental plane, and the maximum is again formed in the fundamental plane. In this case, when the centre of the shadow is at c, the point b is *on* the meridian, and when the centre is at d, the point a is also in this case toward the east, from the meridian. The node, it is seen, is always distant from the meridian, in the direction of the end of the path which is farthest from the pole.

But the degrees of longitude are so small where this point is formed that the longitude of it may, without error, be taken as that of the meridian of the sun at noon.

$$\mu_1 \quad \text{or} \quad \mu_1 \pm 180°.$$

The maximum curve always lies between the rising and setting curve and the pole. And when the pole is elevated, as in CHAUVENET'S example, and the shadow passes over and includes the pole, the maximum lies near the node. But when the pole is depressed and the shadow does not reach the pole, the maximum lies much farther from the node than in the preceding case—sometimes as much as one or two degrees. This is quite noticeable in the eclipses of 1898, July 18; and 1899, January 11.

78. *Greatest Eclipse.*—CHAUVENET'S chapter has no mention of this; but under these words, the data for this phase, for every par-

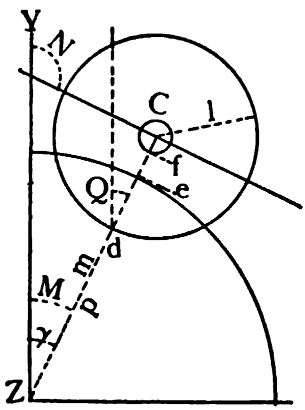
tial eclipse, is given in the *Nautical Almanac*. In Art. 53 we gave the formulæ for the criterion, and in Art. 52 for the time, as follows:

$$\tau = -\frac{m_0}{n}\cos(M_0 - N) \tag{110}$$

$$T = T_0 + \tau \tag{111}$$

In a partial eclipse the extreme times will lie somewhat closer together than in a larger eclipse. The mean of the two terms of the beginning and ending should be gotten closely to at least four decimals of a minute, on account of its effect on the longitude. The point on the earth's surface from which the greatest eclipse is visible must lie in the fundamental plane, since that part of the earth is nearest to the centre of the shadow; and, as its name implies, must be a point of the maximum curve. Hence, the formulæ in this section are applicable to this special case.

FIG. 10.



In determining the time as above by the condition that $m = p + l$ is a minimum, this distance is CZ (Fig. 10), which line is consequently perpendicular to the path, and the following changes result:

$$\text{In 57} \quad \frac{p+l}{n}\cos\psi = 0 \qquad \text{or} \qquad \psi = \pm 90° \tag{112}$$

This equation, as we saw in Art. 58, is independent of the extreme times, and is general for the whole eclipse.

$$\text{In 59} \quad \gamma = N + \psi = N \pm 90° \tag{113}$$

If x and y are taken at the time of the middle of the eclipse, we must have

$$M = \gamma \tag{114}$$

$$= N \pm 90° \tag{115}$$

The general condition for the maximum curve (CHAUVENET, i., p. 476, Art. 309) is

$$\sin(Q - E) = 0$$

that is, $$Q = E, \qquad \text{or} \qquad Q = E + 180° \tag{116}$$

At the point d, whence the shadow crosses the line CZ, we have the angle of position, Q, of the centre of shadow,

$$Q = M.$$

Whence $\qquad M = E$, in northern hemisphere.

$\qquad\qquad M = 180° + E$, in southern hemisphere.

Equation 110 may be put under another form, giving τ_1 from the time of conjunction. At this time $M_0 = 0$, $m_0 = y_0$, and we also have generally $n \cos N = y'$; substituting these values, we obtain the time of greatest eclipse from the time of conjunction

$$\tau_1 = -\frac{y_0 y'}{n^2} \tag{117}$$

which can be used as a check. n may be taken from the computation for the extreme times, and the other quantities are in the eclipse tables.

The above transformations are given for the reader's information, that he may see the changes in the several quantities, rather than for practical use. E, however, is theoretically correct for giving the latitudes and longitudes, but as it is given only to the tenth of a minute in the table, M should be preferably gotten from x and y, especially as m is required for the magnitude; but if M and m are not computed, m may be interpolated from the rising and setting curve.

The formulæ for greatest eclipse will then be, approximately,

$$M = E, \quad \text{or} \quad M = E + 180°, \text{ interpolated from the tables.} \tag{118}$$

$$m \qquad \text{interpolated from R. and S. curve.} \tag{119}$$

Or, more accurately, for the time T equations (110) and (111).

$$\left.\begin{aligned} m \sin M &= x, \\ m \cos M &= y. \end{aligned}\right\} \tag{120}$$

$$\tan \gamma' = \rho_1 \tan \gamma = \rho_1 \tan M. \tag{121}$$

$$\left.\begin{aligned} \cos \varphi_1 \sin \vartheta &= \sin \gamma', \\ \cos \varphi_1 \cos \vartheta &= -\cos \gamma' \sin d_1, \\ \sin \varphi_1 &= \cos \gamma' \cos d_1. \end{aligned}\right\} \tag{122}$$

$$\tan \varphi = \frac{\tan \varphi_1}{\sqrt{1 - e^2}}. \tag{123}$$

$$\omega = \mu_1 - \vartheta. \tag{124}$$

Compute with five-place logarithms to seconds of arc.

The reader will notice that $M = E$ is geometrically shown in Fig. 3, the quantities forming the miniature path being such that this is

parallel to the real path; and when this figure is transferred from Z to F, the two perpendiculars to these lines form one straight line.

79. *Magnitude.*—The following transformations take place under the conditions of the previous article:

In (95) since $M = E$, $\sin \psi = 0$.
And (97) becomes (Fig. 10) $\Delta = m - p = ZC - Ze = Ce$. (125)

The distance the point e is immersed in the shadow is

$$de = l - \Delta. \tag{126}$$

The point f of the umbral cone is on the dividing line where the sun is wholly obscured, so that the magnitude of the eclipse is the ratio $de : df$; and as Cf, the radius of the umbral shadow, is negative for a total eclipse, df will be the algebraic sum $l + l_1$. Hence, we have the formula for the degree of obscuration, D, of CHAUVENET, generally called the magnitude in the *Almanac*.

$$\Delta = m - p. \tag{127}$$

$$M = \frac{l - \Delta}{l + l_1}. \tag{128}$$

The above computation gives m; and p, being the radius of the earth, can be taken from Table IV. Reduce all the quantities to natural numbers for these equations. These formulæ give a decimal of the sun's diameter taken as unity. Because when M becomes equal to unity, the sun is wholly obscured on the earth at e (Fig. 10). The magnitude is given to three decimals in the *Almanac*.

SECTION VIII.

OUTLINE OF THE SHADOW.

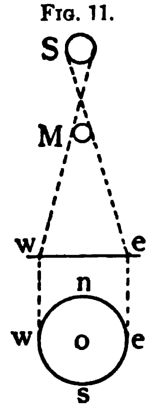
80. THIS curve shows the outline of the penumbral shadow upon the earth at any moment. The times are here assumed at pleasure for the centre of the circle, and are generally taken for each integral hour successively, giving from three to five curves. They are sometimes called *Hour Circles* (Figs. 2, 3, and 8).

The Angle Q.—There is a mistake in CHAUVENET'S text, page

447, Art. 292, and also in Fig. 42, that greatly obscures his explanation of the angle Q. In the figure the line M_1C_1 should not pass through the centre O. M is the centre of the circle of shadow, and C_1 is *any* point on this circle. The whole circle, or at least an arc, might well have been drawn in perspective. The triangle M_1C_1N should be moved a little to the right, and M_1C_1 should then meet the axis in some point between O and Y. The figure as drawn is correct for *one* position of C_1, but is not general. When changed as here suggested, the text will not apply to the figure.

A modification of CHAUVENET's figure 39, page 440, will explain this angle. In Fig. 11, let $w\ e$ be the fundamental plane and the circle below, the outline of the shadow on the plane. The axis of the cone and shadow, the line joining the centres of the sun and moon, is perpendicular to the fundamental plane at all times. Suppose a plane to be passed through the axis of the cone, it will pass through the centres of the sun and moon, and also through their point of visual contact. If it also passes through the point w, an observer at this point will see the *western* limb of the moon tangent to the *eastern* limb of the sun, the moon being due east of the sun. An observer at S will see the moon likewise due north of the sun. The observer having moved 90°, the moon has apparently done the same; and generally, from whatever position the observer is at on this circle of shadow, the moon will be seen on the opposite side of the sun's disk.

FIG. 11.



Q is the position angle of the centre of the shadow from any point on the circle. At S this angle is zero, and the moon being seen due north of the sun, the position angle of the moon on the sun's disk is also 0. At w the angle is 90°; the moon being seen due east of the sun, its position angle on the sun's disk is also 90°. And generally, Q, the position of the centre of the shadow, is at all times equal to the position angle of the moon on the sun's disk. The angle is measured from the north point of the sun's disk and from the axis of Y on the earth toward the east as positive.

In Art. 298 of CHAUVENET's *Astronomy*, page 459, occurs the following equation, not numbered, but which we will refer to as equation (a):

$$\zeta_1^2 = \cos^2\beta - 2\,i\zeta_1 \sin\beta \cos(Q - \gamma) - (i\zeta_1)^2 \qquad (a)$$

The last term, being the square of a very small quantity, is omitted.

By transposing the term containing ζ_1 to the left member, completing the square, and then extracting the root of both members, we obtain—

$$\zeta_1 = \pm \cos\beta - i \sin\beta \cos(Q - \gamma) \qquad (b)$$

But we may obtain this under the form that CHAUVENET has given by replacing ζ_1 in the second number of equation (a) by $\cos\beta$, since $\cos\beta = \zeta$ exactly, and $= \zeta_1$ nearly; then completing the square as above and taking the root, we obtain

$$\zeta_1 = \pm [\cos\beta - i \sin\beta \cos(Q - \gamma)] \qquad (c)$$

As $(Q - \gamma)$ gives any point of the outline of shadow, if we let it represent two points diametrically opposite one another, it will take the double sign $\pm$, which will then reduce equation (b) to the form (c). CHAUVENET makes use of the latter equation, but it is unimportant, since it is used only for ε and to determine the sign of ζ_1; and this he ascertains, in the next following paragraph, simply by a process of reasoning.

81. *General Formulæ.*—

$$\left.\begin{aligned} \sin\beta \sin\gamma &= x - l \sin Q = a \\ \sin\beta \cos\gamma &= \frac{y}{\rho_1} - \frac{l \cos Q}{\rho_1} = b \end{aligned}\right\} \qquad (129)$$

$$\text{I.} \left\{ \begin{aligned} \varepsilon &= \frac{i \cos(Q - \gamma)^*}{\sin 1''} && (130) \\ \zeta_1 &= \cos(\beta + \varepsilon) && (131) \\ \left.\begin{aligned} \xi &= a + i\zeta_1 \sin Q \\ \eta_1 &= b + l\zeta_1 \cos Q \end{aligned}\right\} & && (132) \end{aligned}\right.$$

For greater accuracy, instead of the previous group, take the following:

$$\text{II.} \left\{ \begin{aligned} \varepsilon &= \frac{i \cos(Q - \gamma)}{\sin 1''} && (133) \\ \varepsilon' &= (d_1 - d_2) \cos\gamma && (134) \\ \left.\begin{aligned} \sin\beta' \sin\gamma' &= a + i\rho_2 \cos(\beta + \varepsilon + \varepsilon') \sin Q = \xi \\ \sin\beta' \cos\gamma' &= b + \frac{i\rho_2 \cos(\beta + \varepsilon + \varepsilon') \cos Q}{\rho_1} = \eta_1 \end{aligned}\right\} & && (135) \\ \zeta_1 &= \cos\beta' && (136) \end{aligned}\right.$$

* This quantity may be approximately taken from Table X. An example of its use is given in Section X.

Whichever one of the two above groups has been used, then proceed as follows:

$$\left.\begin{aligned} c \sin C &= \eta_1 \\ c \cos C &= \zeta_1 \end{aligned}\right\} \tag{137}$$

$$\left.\begin{aligned} \cos \varphi_1 \sin \vartheta &= \xi \\ \cos \varphi_1 \cos \vartheta &= c \cos (C + d_1) \\ \sin \varphi_1 &= c \sin (C + d_1) \end{aligned}\right\} \tag{138}$$

$$\tan \varphi = \frac{\tan \varphi_1}{\sqrt{1 - e^2}} \tag{139}$$

$$\omega = \mu_1 - \vartheta \tag{140}$$

These formulæ are rigorous and will bear five-place logarithms and close work; the computer may take his choice of the two groups, which one to use, according to the accuracy required. Formulæ (134) and (135), it will be noticed, require the quantities d_2 and ρ_2, which are not usually computed for an eclipse. They are gotten from formula (51), Art. 34. The equations are to be computed for a series of assumed values of Q, so that the first thing to be done is to find the limits of Q. The time for which the outline is to be computed must, of course, be previously assumed. The simplest way to find Q is from a plot of the eclipse similar to these here given. Thus, in Fig. 8, Plate V., the centre of the 10^h curve bears from the southerly end about $+ 20°$, and the points extend round the circle to $+ 220°$, in which direction the centre bears from the northerly end. Then the series of values is to be assumed every 5° or 10°, as circumstances may require. For the convenience of the computer Table IX. is appended, giving the sines and cosines for every 5° of the circle. The outlines for 8^h and 9^h (Fig. 8) are entire, and Q will give values throughout the whole circle. The small quantity ε, formula (130), can be taken approximately from Table X. If a value should be assumed for Q, for which there is no point of the curve, it will show itself by giving a value of $\sin \beta >$ unity. $\sin \beta$ is always positive, also $\cos \beta$, as the latter is the height of the point on the earth's surface *above* the fundamental plane; and as there can be no eclipse *below* the plane, β will be between 0° and $+ 90°$. The angle C is always less than 90°, but may be either + or —.

82. Formulæ of such rigor as these above given will seldom be required for this curve, though it is the most important of all the penumbral curves. The only case in which these accurate formulæ would be required is probably a drawing on an enlarged scale show-

ing the path of the shadow over a tract of country. The outline curves in this case, computed for short intervals of time, say 5 or 10 minutes, or closer perhaps, will serve to warn observers of the approximate times of beginning and ending, and that is all he requires.

83. The following criterion is given by CHAUVENET, though it will seldom or never be required by the practical computer:

$$\left.\begin{array}{l} e \sin (Q - E) < \zeta_1 f \sin Q, \text{ Eclipse is beginning,} \\ e \sin (Q - E) > \zeta_1 f \sin Q, \text{ Eclipse is ending.} \end{array}\right\} \quad (141)$$

No example of the use of these formulæ for outline is appended, for the reasons given in the next paragraph.

84. *Approximate Formulæ.*—The formulæ above given by CHAUVENET, being carried to their utmost exactness, the reader may perhaps expect some suggestion as to approximate formulæ; but this is deferred to Section X., on the Northern and Southern Limiting Curves, in which, when the angle Q is determined (which is here assumed), the formulæ are the same precisely. And as that curve is of much less importance, it is a suitable place for repeating these formulæ with sufficient exactness for ordinary use.

85. *Geometrical Illustration.*—In Fig. 8, Plate V., the coördinates of the centre, T_0, of the 9^h curve are x and y. The coördinates of the point m, referred to the centre of the shadow, are $l \sin Q$ and $l \cos Q$. Hence, the coördinates of this point referred to the axes are $x - l \sin Q$ and $y - l \cos Q$.

Also the same point referred to the axes by its bearing γ from CZ, and distance Zm, which is here called $\sin \beta$, are $\sin \beta$, $\sin \gamma$, and $\sin \beta \cos \gamma_1$. Hence, we have

$$\sin \beta \sin \gamma = x - l \sin Q = a = \xi$$
$$\sin \beta \cos \gamma = y - l \cos Q = b = \eta_1$$

It follows from this that the third coördinate is $\zeta_1 = \cos \beta$; of which the reader may not need an explanation; however, it is as follows: It results upon the general equation of the sphere (CHAUVENET, Art. 298, p. 458, equation (499)).

$$\xi^2 + \eta_1^2 + \zeta_1^2 = 1 \quad (142)$$

substituting in this the values of ξ and η_1 from the previous equation, $\sin^2\gamma + \cos^2\gamma$ cancels out, and we have left

$$\sin^2\beta + \zeta_1^2 = 1$$

whence $$\zeta_1^2 = 1 - \sin^2\beta \quad \text{or} \quad \zeta_1 = \cos\beta.$$

The small term ϵ is added to β, to take account of the diminished radius of the penumbral shadow as we approach the fundamental plane in a total eclipse. In the equation

$$\epsilon = \frac{i\cos(Q-\gamma)}{\sin 1''}$$

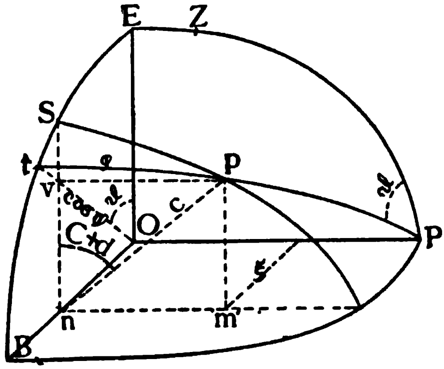
$i = \tan f$, the angle of convergence of the cone; $\tan f$ is the amount of this at the height ζ_1 above the fundamental plane. And being added to the angle β, when $\beta = 0°$, ζ is at the point Z (Fig. 8), and its value not changed, because the surface of the sphere is parallel to the fundamental plane. When $\beta = 90$, it affects the angle, but not the cosine; and ζ is not affected at this point, because the divergence of the cone is 0. The correction, it is seen, very well answers its purpose when applied to the angle β, rather than in any other way. Moreover, at the point c (Fig. 13), where $Q = \gamma$, ϵ will be positive, which *increases* the angle, but *diminishes* ζ; which is correct, as the convergence of the cone diminishes the radius toward the centre of the cone. At the point d, where $Q = 180 + \gamma$, the reverse is the case. The correction is not quite exact at intermediate points, so that a second quantity is added in the rigorous equations.

86. The geographical position of the point is completely determined by these three coördinates. In Fig. 8 let m be the point on the 9^h curve. Pass a vertical plane through this point perpendicular to the axis of X at n, and revolve it down in the principal plane; m and n being in the axis of revolution, will remain unchanged; the points of the sphere vertically above them will fall at p and q respectively on the semicircle of which mn produced is the diameter; the point r of the equator will fall at s.

Then
$$pnq = C$$
$$pn = c$$
$$qns = d \quad \text{or} \quad d_1$$
$$pns = C + d_1$$

And the point m is now referred to the equator instead of the plane XZ, so that the latitude now soon follows.

FIG. 12.



The relations of the quantities following being in space, are more clearly seen from Fig. 12, annexed, which is a perspective view of the eighth part of the sphere. The reference letters being the same as in Fig. 8, Plate V., the points can readily be identified.

EB is the equator. P is the pole.

$pns = (C + d)$.

$pn = c$.

$pm' = c \cos (C + d_1)$.

$m'n = c \sin (C + d_1)$.

Also Ppt is the meridian of the point p.

$pt = \varphi$, the arc of latitude.

$Ept = \vartheta$, the hour angle.

Then $Ov = \cos \varphi$.

$On = \cos \varphi \sin \vartheta = \xi$.

$vn = pm' = \cos \varphi \cos \vartheta = c \cos (C + d_1)$.

$pv = m'n = \sin \varphi \qquad = c \sin (C + d_1)$.

Small quantities must necessarily be omitted in the above illustrations, and $d = d_1$, so far as can be seen on a drawing of the size of Fig. 8.

87. *Dr. Hill's Formulæ for Outline Curves.*—The formulæ are similar to those prepared by him for projecting the Transit of Venus in 1874.*

NOTATION FOR THIS ARTICLE.

h = altitude of the sun above the horizon, nearly.
= altitude of the point of contact above the fundamental plane. h is always less than 90°.

θ = parallactic angle of point of contact.
= where projected on the fundamental plane, the angle of point of contact from the axis of Y.

ρ = radius of the terrestrial spheroid.

* Papers on the Transit of Venus in 1874. Report of, by Mr. George W. Hill, Washington, 1872.

The fundamental equation in the theory of eclipses (CHAUVENET, i., p. 449, equation (490)) is,

$$(l - i\zeta)^2 = (x - \xi)^2 + (y - \eta)^2 \tag{143}$$

In the small term place ζ a mean value $= \frac{1}{2}$.

$$(l - \tfrac{1}{2}i)^2 = x^2 + y^2 - 2(x\xi + y\eta) + \xi^2 + \eta^2$$

Place

$$x = m \sin M \qquad y = m \cos M$$
$$(l - \tfrac{1}{2}i)^2 = m - 2(x\xi + y\eta) + \xi^2 + \eta^2 \tag{144}$$

Now h being the altitude of the place above the fundamental plane, we have

$$\zeta = \rho \sin h$$
$$\xi^2 + \eta^2 = \rho^2 \cos^2 h$$
$$\xi^2 + \eta^2 + \zeta^2 = \rho^2 = 1 \text{ nearly}$$
$$(l - \tfrac{1}{2} i)^2 = m^2 - 2(x\xi + y\eta) + \rho^2 \cos^2 h$$
$$x\xi + y\eta = \frac{m^2 - (l - \frac{1}{2} i)^2 + \rho^2 \cos^2 h}{2}$$

But

$$\xi = \rho \cos h \sin \theta$$
$$\eta = \rho \cos h \cos \theta$$

And place $\rho = 1$

and

$$x\xi + y\eta = m \sin M \cdot \rho \cos h \sin \theta + m \cos M \cdot \rho \cos h \cos \theta$$
$$= m \cos h \cos (\theta - M)$$
$$\cos(\theta - M) = \frac{m^2 - l^2 + \cos^2 h}{2m \cos h}$$

the small term $\frac{1}{2} i$ in the above equations being omitted. This equation completely determines the point. The sign of $\cos(\theta - M)$ is determined; it is usually positive; but when the curve goes over the zenith for some few points the angle is obtuse. $(\theta - M)$ has two values, positive and negative. It is analogous to λ in CHAUVENET's formulæ here given, No. 78 of the Rising and Setting formulæ. It is the angle at the centre of the sphere to any point of the curve. In Fig. 13, A, if $(\theta - M) = 0$, $\cos(\theta - M) = \pm 1$, then equation (147) gives

$$\pm (m \pm \cos h) = \pm l.$$

When the outline touches the horizon (Fig. 13, A), which is the most usual case,

$$\cos h = m - l$$

the superior limit a, and there is no inferior limit, since it falls off the sphere.

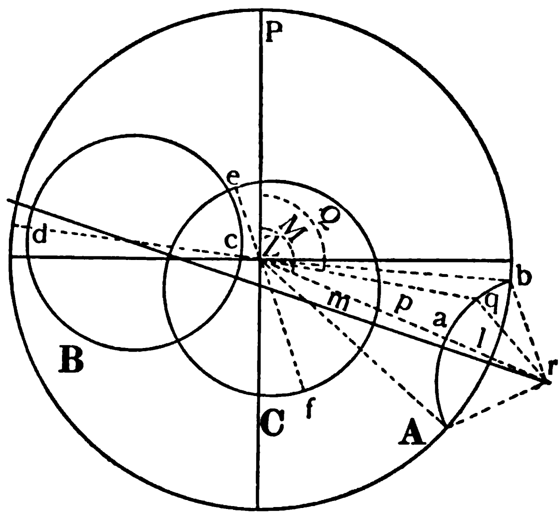
When the circle does not touch the horizon (Fig. 13, *B*),

$$\cos h = m - l, \text{ the superior limit } c$$
$$\cos h = m + l, \text{ the inferior limit } d$$

When the curve passes over the zenith (Fig. 13, *C*),

$$\cos h = m - l, \text{ still the superior limit } e$$
$$\cos h = m + l, \text{ the inferior limit } f$$

FIG. 13.



In the latter case, however, $m - l$ is negative, and for a portion of the curve adjacent, $\theta - M > 90°$, and becomes $= 180°$ at the point *e*.

88. Hence we have for practical use,
Limits of *h* when the curve touches the horizon,

$$\left.\begin{array}{l} h = 0, \quad \text{or} \quad \cos h = 1 \\ \qquad\qquad\quad \cos h = m \sim l \\ \text{When the curve does not touch the horizon,} \\ \qquad\qquad\quad \cos h = m + l \\ \qquad\qquad\quad \cos h = m \sim l \end{array}\right\} \tag{145}$$

$$\left.\begin{array}{l} m \sin M = x \\ m \cos M = y \end{array}\right\} \tag{146}$$

$$\cos(\theta - M) = \frac{m^2 - l^2 + \cos^2 h}{2m} \sec h. \tag{147}$$

And for the geographical positions,

$$\left.\begin{aligned} \cos\varphi\sin\vartheta &= \cos h\sin\theta \\ \cos\varphi\cos\vartheta &= \cos d\sin h - \sin d\cos h\cos\theta \\ \sin\varphi &= \sin d\sin h + \cos d\cos h\cos\theta \end{aligned}\right\} \qquad (148)$$

$$\omega = \mu_1 - \vartheta$$

As d varies but a few minutes during an eclipse, it may be taken as constant, as also l. M and m may be taken from the rising and setting curve for the eclipse hours.

It will be noticed that the following are constant for all eclipses: log sin h, log cos h, log sec h, and $\cos^2 h$ natural numbers. They may be prepared beforehand, and are given in Table XI. for every five degrees.

The following are constant for one eclipse:

$$\log\sin d, \qquad \log\cos d, \qquad l^2 \text{ in numbers};$$

also the factors,

$$(1) = \log(\cos d\sin h), \qquad (2) = \log(\sin d\cos h),$$
$$(3) = \log(\sin d\sin h), \qquad (4) = \log(\cos d\cos h).$$

And the following are constants for each curve:

$$M, \quad \mu_1, \quad m^2 \text{ in numbers} \quad \text{or } (m^2 - l^2), \quad (m - l) \quad \log m.$$

These can be prepared beforehand on the lower edge of a slip of paper—the general constants for every 5° or 10° of h.

Compute with four-place logarithms to minutes for the eclipse hours. $(\theta - M)$, having the double sign, gives two points to the curve. The inferior limit gives but one point for which $\theta = M$. The superior limit when it exists also gives but one point, for which $\theta = M + 180°$. Instead of finding tangent φ, it is better to find the angle from its sine and cosine, and their agreement with the same angle is a considerable check upon the accuracy of the work. If the computer has confidence in his work, he may find φ from its cosine alone when the point is not too near the equator. If the sine and cosine agree within 10 minutes, the point may be passed as correct, or at least until plotted on the chart, for the ink line to be drawn through this point will be about 10 or 15 minutes of arc in width. A plot of the eclipse similar to those here given will be found very useful in roughly checking the work, or deciding the quadrant to be taken. If a scale be made of cosines to the radius of the sphere, any point can at once be located on the hour circles.

89. *Geometrical Illustration.*—Not much is required under this heading. Equation (147) is, after all the transformations, very simple. In the common formula for a plain oblique triangle,

$$a^2 = b^2 + c^2 - 2bc \cos A$$

or
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

If we apply this to the triangle, Zrq (Fig. 13), which lies in the fundamental plaue, calling A the angle at Z, equation (147) results at once, for $Zq = \cos h$, $rq = l$, $Zr = m$. The angle $PZr = M$ is known; $PZq = \theta$ is required, which is given by $\theta - M = qZr = A$ of the above general formula.

To proceed further: if in the general equation $\sin \frac{1}{2}a = \sqrt{\frac{1-\cos a}{2}}$, we substitute the value of $\cos(\theta - M)$ from Dr. Hill's formula (147), we get, after reducing,

$$\sin \tfrac{1}{2}a = \sqrt{\frac{l^2 - (m-p)^2}{4pm}} = \sqrt{\frac{[l-(m-p)]\,[l+(m-p)]}{4pm}}$$

which is CHAUVENET'S equation (78) deduced from Dr. Hill's. Equations (148) are of the usual forms, the point p being referred first to the fundamental plane and axes, then the axes revolved through the angle d, which is the inclination of the equator to the plane XZ.

In the example following are a few references to Fig. 8 in explanation. The reader will readily see that all these quantities, lines, and angles can be measured on the drawings given to scale in the plates.

90. *Example.*—Outline by Dr. Hill's Formulæ. The several constants for this eclipse and for the 9^h curve are given with the example. It will be necessary almost to write these off on a slip of paper, and to follow the formulæ, to understand the example, for they are not given there. The example stands just as it does on my computing sheet, except that it should form *one* column. Constants are written *once*, and I avoid as much as possible repeating them. In Fig. 8, Plate 5, if a circle of altitude be drawn whose radius is $\cos h$, it will strike the 9^h outline in two points, i and j, which are those of the example. The references (2) and (4) in the precept to the example refer to those constants given in the margin. (1) and (3) are to be used from a slip of paper. The natural numbers are used for $m^2 - l^2$,

and this, added to $\cos h$, natural numbers, gives the numerator; then pass to logarithms $(\theta - M)$. θ and M can be measured on Fig. 8, $\theta = CZi$ and CZj, and corresponds to γ of CHAUVENET's formulæ. The quantities in equation M and $\log m$ are taken from the rising and setting curve generally, but the example given was not the 9^h curve. However, as this is the epoch hour, these quantities may be found in Article 54, Extreme Times Generally.

OUTLINE CURVES, HILL'S FORMULÆ.

EXAMPLE, TOTAL ECLIPSE, 1904, SEPT. 9.

Constants for all curves for this eclipse, intermediate values being omitted.

h.	20	40	60	80
(1)	+ 9.5323	9.8063	9.9357	+ 9.9916
(2)	+ 8.9342	8.8455	8.6602	+ 8.2009
(3)	+ 8.4953	8.7693	8.8987	+ 8.9546
(4)	+ 9.9712	9.8825	9.6972	+ 9.2379

Constants for this curve.

μ_1	135° 42′	9^h curve
M	154 11	for $h = 60°$
$\log \frac{1}{2m}$	0.3515	

	h.	60
$(m^2 - l^2)$ — 0.2342	(135) numerator	+ 0.0158
	log	+ 8.1987
	$\cos(\theta - M)$	+ 8.8512
	$(\theta - M)$	∓ 85 56

h.	60	60
(136) θ	+ 68 15	+ 240 7
sin	+ 9.9679	— 9.9380
cos	+ 9.5689	— 9.6974
$\cos\phi \sin\vartheta$	9.6669	— 9.6370
(2) $\cos\theta$	+ 8.2291	— 8.3576
$l - l$	B 1.7066	A 1.5781
	A 1.6980	B 1.5894
$\cos\phi \cos\vartheta$	+ 9.9271	+ 9.9470
$\tan\vartheta$	9.7398	9.6900
ϑ	+ 28 47	— 26 6
cos	9.9427	9.9533
$\cos\phi$	+ 9.9844	+ 9.9937
(4) $\cos\theta$	+ 9.2661	— 9.3946
$l - l$	A 0.3674	B 0.4959
	B 0.5225	A 0.3289
$\sin\phi$	+ 9.4212	— 9.2276
ϕ	+ 15 17	9 48
(137) ω	+ 106 55	+ 161 48

It may be added that the omission of $i\zeta$ or $\frac{1}{2}i$ in equation (147) has exactly the same effect as omitting the quantity ϵ (equation (130)) from the approximate formulæ referred to in Article 84, for they both take account of the decreased radius of the hour circles as the points rise above the fundamental plane.

The Rising and Setting Curve.—Article 64 is derived from the above formulæ simply by the condition that $h = 0$. All the formulæ are simplified by the point being in the fundamental plane, and $(1 - l^2)$ is the only constant.

SECTION IX.

EXTREME TIMES, NORTHERN AND SOUTHERN LIMITS OF PENUMBRA.

91. In this section Chauvenet remarks that he has not followed Bessel, but has given formulæ better suited to practical use. *The formulæ in full* are as follows:

$$\left.\begin{aligned} m \sin M &= x_0 \mp l \sin E_0 \\ m \cos M &= y_0 \mp l \cos E_0 \end{aligned}\right\} \qquad (149)$$

$$\left.\begin{aligned} n \sin N &= x_0' \mp \frac{l}{e} b_0'' \\ n \cos N &= y_0' \mp \frac{l}{e} c_0'' \end{aligned}\right\} \qquad (150)$$

$$\sin \psi = m \sin (M - N) \qquad (151)$$

$$\tau = \frac{\cos \psi}{n} - \frac{m \cos (M - N)}{n} \qquad (152)$$

$$T = T_0 + \tau \qquad (153)$$

$$\left.\begin{aligned} \gamma &= N + \psi \\ \tan \gamma' &= \rho_1 \tan \gamma \end{aligned}\right\} \qquad (154)$$

$$\left.\begin{aligned} \cos \varphi_1 \sin \vartheta &= \sin \gamma' \\ \cos \varphi_1 \cos \vartheta &= - \cos \gamma' \sin d_1 \\ \sin \varphi_1 &= \cos \gamma' \cos d_1 \end{aligned}\right\} \qquad (155)$$

$$\tan \varphi = \frac{\tan \varphi_1}{\sqrt{1 - e^2}} \qquad (156)$$

$$\omega = \mu_1 - \vartheta \qquad (157)$$

$$\text{Taking } E \text{ acute, } \left\{\begin{array}{l} \text{For Northern Limits, take the lower sign.} \\ \text{For Southern \quad `` \quad `` \quad upper \quad ``} \end{array}\right. \qquad (158)$$

Compute with four-place logarithms to minutes, and with but one approximation. The compression may be neglected in this curve; then

$$\gamma = \gamma', \qquad \rho_1 = 1, \qquad \varphi = \varphi_1, \qquad e = 0.$$

For these points $Q = E$, which is here taken as acute, which gives rise to the double sign, for in fact E has two values differing by 180°.

The quantities with subscript zero are to be taken for the epoch hour; l also may be taken as constant, and if the above approximations are made, d may be used instead of d_1 and taken as constant at the epoch hour. ψ, as in similar equations, has the two values, acute and obtuse, giving the $+$ and $-$ values to the cosine.

The equations for N are the hourly motions of the equations for M. If we substitute in the first equation $e \sin E = b'$, we have in the right member $x_0 \mp l \frac{b'}{e}$. Now the hourly motion of x is x', and of b' it is b'', which gives the first equation for N, e being taken as constant.

Chauvenet states in Article 313, at the bottom of page 485, that e may be taken as constant; it is because it has but little effect, occurring only in a very small term. If e is interpolated for the two times of beginning and ending, it will be found that the two values are quite or nearly the same, since e has a maximum or minimum at the middle of the eclipse.

These times are not of much importance, no figures are given, and they are used merely for the geographical positions to check the other curves. φ should be looked out from both sine and cosine as a check.

The middle of the eclipse for these times is given by the formula explained in Article 52.

$$\tau = -\frac{m \sin (M - N)}{n} \tag{159}$$

$$T = T_0 - \tau \tag{160}$$

This may not agree exactly with that found for beginning and ending, but should not vary much from it. They agree very closely in this eclipse, but it is a mere chance and nothing more.

The formulæ are so similar to several of those already given and explained that they need no further comments, especially as the extreme points have already been explained in Article 72 of the maximum curve.

92. *Example.*—This comprises the whole work for the times, only one approximation being made; but the latitudes and longitudes are omitted, since the formulæ are the same as those used in several examples on former pages, especially for the outline curves. Log $(1 : e)$ in this example is 0.2364. The formulæ numerically are the same for beginning and ending except when the double sign gives two values. It is well to place the sign at the head of the columns; for the north points the terms of the first two formulæ are added, and for the south points, subtracted. Addition and subtraction logarithms are very convenient here, since the difference of the logarithms here noted as $l - l$ is the same for both, being called A if the terms are numerically added, and B if they are numerically subtracted; $\cos \psi$ is minus for beginning and plus for ending.

Extreme Times, N. and S. Limits, Penumbra.

Example, Total Eclipse, September 9.

		N. (+).	S. (−).
Formula 149	$\log l$	+9.7264	
	$\sin E_0$	+9.4803	
	$\cos E_0$	+9.9792	
149	x	+8.9866	
	$l \sin E_0$	+9.2067	
	$l - l$	A 0.2201	B
		B 0.4249	A 9.8195
	$m \sin M$	+9.4115	−8.8061
149	y_0	−9.3018	
	$l \cos E_0$	+9.7056	
	$l - l$	B 0.4038	A
		A 0.1858	B 0.5483
	$m \cos M$	+9.4876	−9.8501
149	$\tan M$	+9.9239	8.9560
	M	+40 1	−174 50
	cos	9.8842	9.9982
	$\log m$	+9.6034	+9.8519

		N. (+).	S. (−).
150	$\log (l:e)$	+9.9628	
$\log (1:e)$ 0.2364	b''	+8.1268	
	c''	−7.6182	
150	x_0'	+9.7465	
	$(l:e)\, b''$	+8.0896	
	$l - l$	A 1.6569	B
		B 95	A 97
	$n \sin N$	+9.7560	+9.7368
150	y_0'	−9.2380	
	$(l:e)\, c''$	−7.5810	
	$l - l$	A 1.6570	B
		B 95	A 97
	$n \cos N$	−9.2475	−9.2283
150	$\tan N$	0.5085	0.5085
	N	+107 14	+107 14
	sin	9.9801	9.9801
	$\log (1:n)$	+0.2241	+0.2433

		On N. Curve.		On S. Curve.		Middle.
151	$M - N$	−67 14		−28.2 4		
	sin ( )	−9.9648		−9.9903		
	cos ( )	+9.5871		+9.3202		
	$\sin \psi$	−9.5682		−9.8422		
	ψ	−158 17	−21 43	−135 57	−44 3	
152	$\cos \psi$	−9.9680	+9.9680	−9.8566	+9.8566	
	log (1)	−0.1921	+0.1921	−0.0999	+0.0999	
	(2)	+9.4146		+9.4154		
	Nos. (1)	−1.5563	+1.5563	−1.2587	+1.2587	
	−(2)	−0.2598	−0.2598	−0.2603	−0.2603	τ −0.260
	τ	−1.816	+1.297	−1.519	+0.998	T 8.740
						T $8^h\ 44^m.4$
153	T	7.184	10.297	7.481	9.998	
		7 11.04	10 17.82	7 28.86	9 59.88	

The several times we have now computed for the eclipse may now be compared in the following manner. The central times can also be included, as here shown, after they have been computed.

The internal contacts, not being computed, are not sufficiently exact to be included with the other quantities. The differences throughout this whole eclipse are much more even than usually found among eclipses.

TOTAL ECLIPSE, 1904, SEPTEMBER 9.

Art. 54, First Contact (Beginning),	6^h	$7^m.8$		
			0 55.2	
" 107, Central Eclipse "	7	3 .0		
			0 8.0	
" 92, Northern Limit "	7	11 .0		
			0 17.9	
" 92, Southern Limit "	7	28 .9		
			1 15.5	
" 107, Middle of the Eclipse,	8	44 .4		
			1 15.5	
" 92, Southern Limit (Ending),	9	59 .9		
			0 17.9	
" 92, Northern Limit "	10	17 .8		
			0 7.9	
" 107, Central Eclipse "	10	25 .7		
			0 55.2	
" 54, Last Contact "	11	20 .9		
" 55, Second Contact (Internal),	7	58.	(by scale)	
" 55, Third Contact "	9	29.	"	

SECTION X.

NORTHERN AND SOUTHERN LIMITING CURVES OF PENUMBRA.

93. CHAUVENET'S formulæ in full are as follows:

For Limits of Q,
$$\tan \nu = \frac{f}{e} \tag{161}$$

$$\tan \psi = \tan (45^\circ + \nu) \tan \tfrac{1}{2} E \tag{162}$$

$$\text{Limits,} \left\{ \begin{array}{ll} \text{S. curve, } Q \text{ between } E & \text{and } \tfrac{1}{2} E + \psi \\ \text{N. curve, } Q \text{ between } 180^\circ + E & \text{and } 180^\circ + \tfrac{1}{2} E + \psi \end{array} \right\} \tag{163}$$

For Q,
$$\left. \begin{array}{l} \sin \beta \sin \gamma = x - l \sin Q = a \\ \sin \beta \cos \gamma = y - l \cos Q = b \end{array} \right\} \tag{164}$$

$$\tan \nu' = \frac{f}{e} \cos \beta \tag{165}$$

$$\tan (Q - \tfrac{1}{2} E) = \tan (45^\circ + \nu') \tan \tfrac{1}{2} E \tag{166}$$

For the geographical positions, equations (129–140) in full, Article 81, for Outline of Shadow. By omitting repetitions, these may be approximated as follows:

$$\left. \begin{array}{l} \sin \beta \sin \gamma = x - l \sin \ Q = \xi \\ \sin \beta \cos \gamma = y - l \cos \ Q = \eta \end{array} \right\} \tag{167}$$

$$\epsilon = \frac{i \cos (Q - \gamma)}{\sin 1''} \quad \text{or Table X.} \tag{168}$$

$$\zeta = \cos (\beta + \epsilon) \tag{169}$$

$$\left. \begin{array}{l} e \sin \ C = \eta \\ e \cos \ C = \zeta \end{array} \right\} \tag{170}$$

$$\left.\begin{aligned} \cos\varphi_1 \sin\vartheta &= \xi \\ \cos\varphi_1 \cos\vartheta &= c\cos(C + d_1) \\ \sin\varphi_1 &= c\sin(C + d_1) \end{aligned}\right\} \tag{171}$$

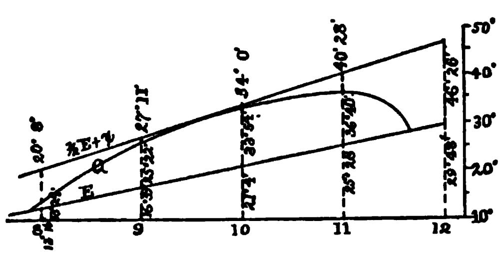
$$\tan\varphi = \frac{\tan\varphi_1}{\sqrt{1 - e^2}} \tag{172}$$

$$\omega = \mu_1 - \vartheta \tag{173}$$

These formulæ can still be greatly simplified, especially the method of finding Q, which comprises nearly half the work of the formulæ; and always seemed to the author very unsatisfactory; though, perhaps, in a strictly mathematical sense it may be the best that can be devised.

94. *The Limits for Q.*—This is illustrated in Fig. 14, the Total Eclipse of 1878, July 29, which is the next successive eclipse in the series to CHAUVENET's example, and thus very similar to the figure which his example would give. The eclipse hours are here numbered; and above the base line, by the scale of degrees in the margin, the angle E is shown; the points being connected give nearly a right line. By formulæ (161) and (162), the superior limit for Q, $\frac{1}{2}E + \psi$, is likewise shown. Between these limits, on the hour lines, we are to assume values for Q for the eclipse hours—a rather uncertain thing to do, when it is considered the true values of Q will lie on the curved line between these limits. Unless we know the form of this curve, two approximations may be necessary with formulæ (164–166) to find Q with sufficient exactness. It is seen that, as CHAUVENET states, $Q = E$, the lower limit, at the beginning and end of the curve; and at the middle approaches or touches the upper limit. In some partial eclipses where this curve does not pass near the zenith of the sphere this curve will not reach the superior limit.

FIG. 14.



95. The vagueness of this method led me to discard it wholly some years ago, and instead I make use of Tables XIII. and XIV., here appended, which will give Q sufficiently exact for the formulæ. Table XIII. is computed by formula (165), and Table XIV. by

(166). The method in detail is to first lay off in Fig. 8, Plate V., the angle E from the centre of each curve; the points where this line crosses the outline curves are marked on the plan by a single dot; $\sin\beta$ is the distance of this point from the centre of the sphere, which is to be carefully measured by scale for each point. This distance is set down in the example under E, Table XIII., with $\log\frac{f}{e}$ (which is constant for one eclipse) gives the angle ν. With this and E, Table XIV. gives Q the first approximation, which is really the second, for E is, in fact, the first.

This value of Q is now laid off by a protractor for each curve and marked on Fig. 8 by two dots. Sin β is to be measured again for the points just found, and, using the tables in the same manner, we get the second approximation for Q, all of which in this example are so near the previous value that we can hardly measure any different value for $\sin\beta$, so we may take the last values of Q as final. In the 10^h curve, which is near the end of the outline limits, it is seen that $Q = E$, nearly, and we can hardly change that value, though the point is not quite at the end of the eclipse. If necessary, Q can now be rigorously computed by formulæ (165–166), on which the tables are founded; but as Q is not needed to be very exact, as CHAUVENET explains in Art. 312, this will not be necessary, especially as we have another check.

An error in Q will shift the point around the circle of shadow, and approximately along the limiting line, and but very little in a lateral direction, so that an error in Q has but little effect in moving the position of this curve.

In the example for the curve we are obliged to compute $\sin\beta$ in order to get $\cos\beta$. Sin β in numbers to three decimals, as computed, is 0.442; whereas in finding Q we have measured for this curve on the plan $\sin\beta = 0.450$, which the succeeding work, by its agreement, proves to be correct, as near as the drawing will give.

I may add that these measurements were made on the drawing of which Fig. 8 is reduced one-half. But in drawings of the same size as Fig. 8 as good results may generally be obtained with care and accuracy in making the plot. It is seen from the example that when there are two limiting curves, $\sin\beta$ must be measured for each, and Q, as resulting, does not differ by exactly 180° for the two points, since Q depends upon $\cos\beta$, which is different for the Northern and for the Southern curves. Sin β must be measured in parts of radius; Fig. 8 being drawn to a scale of 2½ inches as the radius, a scale of

40 parts to the inch will give 100 for radius. Table XIII. shows what little effect log $\frac{f}{e}$ or e has upon the angle Q; which is the reason it may be taken as constant in this curve. E and Q in these tables are taken as acute, being laid off from the axis of Y above and below the centre; but for final values 180° is added for the Northern curve. If E should be negative, it must be laid off toward the *left* hand, which is the negative direction, generally, in the eclipse theory.

96. Example of the Use of Tables XIII. and XIV.

For Finding Q—Total Eclipse, 1904, September 9.

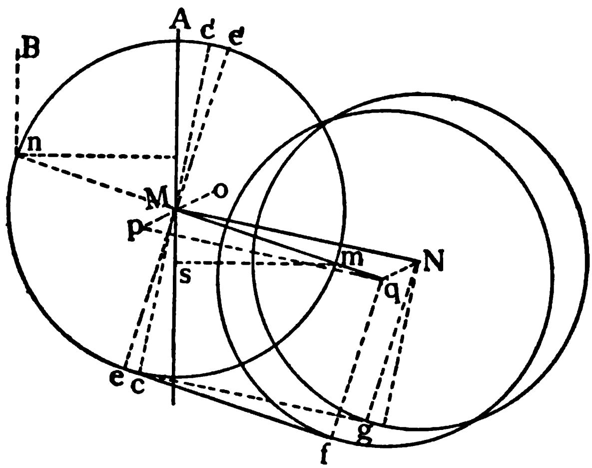
		Northern Curve.			Southern Curve.		
	Hour Curve	8	9	10	8	9	10
	$Q=E$	16 12	17 35	18 58	16 12	17 35	18 58
	$\sin\beta$	0.582	0.408	0.843	0.818	0.718	0.996
	Table XIII. ν	20.1	22.2	13.5	14.5	17.3	2.4
Table XIV.	Q, 1st Approx.	25.2	29.0	24.8	21.7	25.2	19.7
	$\sin\beta$	.515	.450	.887	.835	.690	
	Table XIII. ν	20.9	21.7	11.6	13.8	18.0	
Table XIV.	Q, 2d Approx.	25.7	28.6	23.6	21.2	25.6	
	Values	205 42	208 36	203 36	21 12	25 36	

97. *The Angles Q and E.*—The angle Q being found, we will examine its nature and use and relation to the angle E. Chauvenet remarks of this curve that "it is commonly regarded as one of the most intricate problems in the whole theory of eclipses," the reason for which is the continually changing value which Q takes. In all the preceding extreme times they existed but for the instant of contact—the outline also for the instant of the eclipse hours. The maximum indeed is generated by the motion of the shadow; but Q is known, since it equals E when in the horizon. But here, while the shadow moves in one direction over the fundamental plane, the surface of the earth is also moving, but in quite a different direction; and the motion of the shadow on the earth's surface is somewhat akin to the resultant of these two motions.

We may illustrate this in Fig. 15. Let M be the centre of the shadow moving to N in a given time—say one hour. During this time the surface of the earth at M has moved to o, or p has moved to M and q to N, shortening the path on the earth, since the earth and shadow move in the same general direction; so that Mq is the direction of the centre of the shadow on the earth's surface. If the earth were stationary, the elements of the cone $c\,c'$, perpendicular to MN, would be the points which form the limiting curves; but by

the two motions, $e e'$, perpendicular to Mq, are the tangent points, and ef is the southern limiting curve. We cannot see this motion in Fig. 8, since we have there only the circles centred at M and N on the fundamental plane. If in Fig. 13 we draw from the centre N

FIG. 15.



the line Ng parallel to qf, the points f and g will illustrate the approximations made for finding Q in Fig. 8. The relative duration of the motions in Fig. 13 is approximately the same as in Fig. 8; for example, in the nine-hour curve the earth's surface near the southern curve would move in an ellipse, which, being near the axis of Y, would be nearly perpendicular to it; that is, diverging on the north side of the shadow path, as in Fig. 13, Mo. The angle which Mc makes with the axis of Y is nearly E, and the angle of Me is Q, the angle sought for the limiting curves. This will illustrate these angles, though it may not take account of small terms.

98. It can easily be shown from the formulæ what is the exact difference between the angles Q and E. In Article 300, p. 463, CHAUVENET gives the following equation, No. (513):

$$P' = a' + e \sin (Q - E) - \zeta f \sin (Q - F) \qquad (174)$$

For a maximum curve, $P' = 0$; and in the northern and southern limiting curves (Article 311), the simple contact is the maximum of the eclipse at that point. The equation is simplified by omitting the small quantities a' and F, which gives

$$e \sin (Q - E) = \zeta f \sin Q \qquad (175)$$

which is the fundamental equation for these curves, and from which equations (161–66) are derived.

Developing the above equation,

$$e \sin Q \cos E - e \cos Q \sin E = \zeta f \sin Q$$

By 45,
$$\left.\begin{aligned} e \sin E &= b' \\ e \cos E &= c' \end{aligned}\right\} \tag{176}$$

whence
$$\tan Q = \frac{b'}{c' - \zeta f} = \frac{b'}{c' - f \cos \beta} \tag{177}$$

And by 176,
$$\tan E = \frac{b'}{c'} \tag{178}$$

These equations show the difference of the two angles. By the term ζ entering in the equation for Q, it is seen to vary with the height of the point above the principal plane, and when $\zeta = 0$, $Q = E$. This term being subtracted in the denominator, it follows that $Q > E$.

99. *Approximate Formulæ.*—These do not vary much from those already given, but it may be convenient for use to repeat them.

First find Q by Tables XIII. and XIV., as explained in Article 95 and example Article 96. (179)

$$\left.\begin{aligned} \xi &= \sin \beta \sin \gamma = x - l \sin Q \\ \eta &= \sin \beta \cos \gamma = y - l \cos Q \end{aligned}\right\} \tag{180}$$

$$\zeta = \cos \beta \tag{181}$$

$$\left.\begin{aligned} c \sin C &= \eta \\ c \cos C &= \zeta \end{aligned}\right\} \tag{182}$$

$$\left.\begin{aligned} \cos \varphi \sin \vartheta &= \xi \\ \cos \varphi \cos \vartheta &= c \cos (C + d) \\ \sin \vartheta &= c \sin (C + d) \end{aligned}\right\} \tag{183}$$

$$\omega = \mu_1 - \vartheta \tag{184}$$

Compute with four-place logarithms to minutes for every thirty minutes on the hours and half hours. This curve being chiefly used as a check to the other curves on the chart, is the least important of all the eclipse curves, and no important observations are made near it. To get the intermediate thirty-minute points, arcs of outline curves for the half hours can be drawn in pencil. l may be taken as constant for the whole eclipse, and the compression is here wholly neglected, so that $e = 0$, $\rho_1 = 1$, $\gamma = \gamma'$, $\varphi = \varphi_1$; log e is constant for the eclipse, and Table XIII. shows what small effect it has for the small changes during one eclipse.

100. *Example.*—Q is here found by the method described in Art. 95, in connection with Tables XIII. and XIV. In these approximate formulæ, since ε is omitted, γ is not needed—only its sine or cosine. In the formulæ $\xi^2 + \eta^2 + \zeta^2 = 1$, the general equation of a sphere, which is shown by the example—

$$0.352^2 + 0.2673^2 + 0.897^2 = 0.1239 + 0.714 + 0.8046 = 0.9999.$$

Now, if the small quantity ε, equation (168), is included, this equality will not exist, nor will it exist if η_1 is substituted for η, nor if φ_1 is taken for φ. It is necessary to recompute the coördinates by the group given under the Outline Curves Equations (130–132). Moreover, not only does the equality above not exist, but sin φ and cos φ do not give the same angle. Such great accuracy for this curve is not required, and the only alternative is to omit all these small quantities, which is done in the formulæ of the previous section.

Northern and Southern Limiting Curves.

Example—Total Eclipse, 1904, September 9.

		9^h			
Tables	l	+ 9.7264	(181)	ζ	+ 9.9528
Art. 96.	Q	208 36	(182)	tan C	+ 9.4742
(180)	sin Q	− 9.6801		C	+ 16 36
	cos Q	− 9.9435		cos C	9.9815
	x_0	+ 8.9866		log e	+ 9.9713
	l sin	− 9.4065		$C + d_1$	+ 21 52
	$l - l\,A$	0.4199		sin ( )	+ 9.5711
	B	0.5599		cos ( )	+ 9.9676
	ξ	+ 9.5465	(183)	e cos ( )	+ 9.9389
(180)	y_0	− 9.3018	(183)	tan ϑ	9.6076
	l cos	− 9.6699		ϑ	+ 22 3
	$l - l\,B$	0.3681		cos	9.9670
	A	0.1252		cos ϕ	9.9719
	η	+ 9.4270		sin ϕ	+ 9.5424
	tan γ	0.1195		ϕ	+ 20 24
	sin γ	9.9011			
	sin β	+ 9.6454	(184)	μ_1	135 42
	Numbers	0.442		ω	+ 113 39

It will be noticed that at the end of the first column the quantity sin β is computed, resulting 0.442 in numbers. This serves as a check upon the approximate value 0.450, found in Art. 96, which is about as close as can be measured on a drawing of the size used.

101. *Curves of Any Degree of Obscuration.*—We found for the Maximum Curve the following formula, giving the magnitude

$$M=\frac{L-\Delta}{L+L_1}$$

Δ being the distance the place is immersed in the shadow, if we substitute Δ for l in the formulæ of article 99, and give Δ any value between 0 and unity, the equations (179–184) will give curves north and south of the centre line on which the eclipse has the magnitude assumed for Δ. As L and L_1 are special values depending upon the elevation of the surface of the earth above the fundamental plane, l and l_1 should be substituted for them, and the previous equation becomes

$$\Delta = l - M(l + l_1) \qquad (185)$$

And from (180)

$$\left.\begin{aligned} \xi &= \sin\beta\sin\gamma = x - \Delta\sin Q \\ y &= \sin\beta\cos\gamma = y - \Delta\cos Q \end{aligned}\right\} \qquad (186)$$

Then proceed with the equations (181–184).

The error of using l here instead of L for the given place can never be as much as 0.01 of the sun's disk or $M=0.01$, and usually will be much less.

If M be assumed 0.1, 0.2, 0.3, etc., the series of curves will be approximately parallel to the centre line, the two values of Q giving points north and south of it. The value 0 for Δ would give the northern and southern limiting curves, and the value 1.0 would give the northern and southern limiting curves of the central eclipse. All these lines begin and end on the Maximum Curve in the horizon.

The formulæ would probably have to be computed for every 10 minutes; and l can be taken as constant for the whole eclipse. These curves are not required for the *Nautical Almanac*, nor will they often be required. They, however, again attest the thoroughness of CHAUVENET's treatment of this subject.

In *Harper's Magazine*, vol. iii., p. 239, July, 1851, is a diagram of the United States, showing curves of obscuration of the Solar Eclipse of July 28, 1851. The author read this when a small boy, was interested in it, and his memory now serves him to recall that it illustrates this subject, and to know where to find it.

SECTION XI.

EXTREME TIMES OF CENTRAL ECLIPSE.

102. *Formulæ.*—These are derived by CHAUVENET from those for the penumbra, outline, etc., by the consideration that the radius of the shadow is zero, that is $(l - i\zeta) = 0$; and equation (143), which is the fundamental formula in the whole theory of eclipse, and also the following, derived from it, become zero:

$$\left.\begin{aligned}(l - i\zeta) \sin Q &= x - \xi \\ (l - i\zeta) \cos Q &= y - \eta\end{aligned}\right\} \tag{187}$$

whence

$$x = \xi \quad \text{and} \quad y = \eta \tag{188}$$

For the extreme times, we also have $\zeta_1 = 0$, whence by equation (142)

$$\xi^2 + \eta_1^2 = 1 \tag{189}$$

Also by the above

$$x^2 + y_1^2 = 1 \tag{190}$$

The formulæ for this section are as follows:

$$y_1 = \frac{y}{\rho_1} \qquad\qquad y_1' = \frac{y'}{\rho_1} \tag{191}$$

$$\left.\begin{aligned}M_0 \sin M_0 &= x_0 \\ M_0 \cos M_0 &= y_1\end{aligned}\right\} \tag{192}$$

$$\left.\begin{aligned}n \sin N &= x_0' \\ n \cos N &= y_1'\end{aligned}\right\} \tag{193}$$

$$\sin \psi = m_0 \sin (M_0 - N) \tag{194}$$

$$\tau = \frac{\cos \psi}{n} - \frac{m_0 \cos (M_0 - N)}{n} \tag{195}$$

$$T = T_0 + \tau \tag{196}$$

$$\gamma = N + \psi \tag{197}$$

$$\left.\begin{aligned}\cos \varphi_1 \sin \vartheta &= \sin \gamma \\ \cos \varphi_1 \cos \vartheta &= - \cos \gamma \sin d_1 \\ \sin \varphi_1 &= \cos \gamma \cos d_1\end{aligned}\right\} \tag{198}$$

$$\tan \varphi = \frac{\tan \varphi_1}{\sqrt{1 - e^2}} \tag{199}$$

$$\omega = \mu_1 - \vartheta \tag{200}$$

103. Compute with five-place logarithms to seconds of arc. Two approximations are necessary, computing first for the epoch hour. ψ has, as in similar equations, two values, the obtuse, with negative cosine for beginning; and the acute, with positive cosine for ending; M_0 and $\log m_0$ are computed but once for the epoch hour. For the second approximations of the time take the other quantities for the times just found; x_0' and y_1' are the mean hourly changes. It is generally simpler to take angles which result from these formulæ as

less than 180°, and either positive or negative as the equations denote. Angles, however, formed by addition or subtraction may result greater than 180°. The second approximation of the times will generally vary not over one or two-tenths of a minute. The proportional parts of μ_1 can be taken from Table VIII. The times in the final approximation should be gotten to four or five decimals of a minute, since they affect the longitudes.

The times and geographical positions of the central eclipse may also be computed by the formulæ of Section V. for the extreme times generally, by placing $l = 0$.

104. *Middle of the Eclipse.*—This is similar to preceding cases.

$$\tau = -\frac{m_0 \cos(M_0 - N)}{n} \tag{201}$$

$$T = T_0 + \tau \tag{202}$$

105. *Check Equations.*—These are derived from equation (189), and are similar to those for the extreme times generally. They are computed for the times already found.

$$\left.\begin{aligned} m \sin M &= x \\ m \cos M &= y_1 \end{aligned}\right\} \tag{203}$$

Then

$$M = \gamma \tag{204}$$

$$m = \text{unity}. \tag{205}$$

All the decimals of the times must be used here in order to get x and y_1 sufficiently exact to make M agree with γ. It should agree within three or four seconds; m should be within one or two units of unity, either more or less. Deviations greater than these may perhaps not affect the results. Sine φ_1 and cos φ_1 should be examined to see if they give the same angle within one unit of the last decimal. These check equations are so similar to those in Art. 57 that no example is given.

106. *Geometrical Explanations.*—The points found by the above formulæ are marked K and L on Fig. 2, Plate I., and on Fig. 3, Plate II., giving the times on the path and the geographical positions on the earth's horizon line. The formulæ can be explained and shown in precisely the same manner as in Art. 58 for the extreme times generally, and the geographical positions as in Art. 60, making some few changes in the formulæ; for example, instead of $p + l$ in the former case, we here have $p = m = \rho$, the earth's radius, or $=$ unity, as these differences are too small to be seen on a drawing of the size of the figures here given.

It will be noticed that these points here computed are not points of *contact*, but the passage of the centre line of the umbral cone across the earth. The umbral cone of course forms small circles of rising and setting curves, similar to those of the penumbra, and surrounding the points K and L on the earth's surface; but they are never computed. The points K and L lie on the maximum curve, already computed.

107. *Example.*—The first approximation in one column is here omitted, since the first is precisely like the second, except that the latter is made in two columns, taking out the quantities from the tables for the times given at the head of the columns, which are those of the first approximation. No quantities in the first approximation are used in the second, and doubtless the reader has become so familiar with the methods previously adopted that he will require but few further comments as a guide.

Extreme Times of Central Eclipse.

Example—Total Eclipse, 1904, September 9.

From first Approx. T	7 3.000	10 25.728	Nos. (1)	−1.68900	+1.68896
(192) x_0	+8.98664		—(2)	−0.26073	−0.26077
y_1	−9.30325		τ	−1.94973	+1.42819
tan M_0	9.68339		(196) T	7.05027	10.42819
M_0	+154 14 54			7 3.0162	10 25.6914
sin	9.60797				
	9.37867		(197) γ	+277 53 27	+116 40 35
cos	9.95458		(198) sin γ	−9.99587	+9.95113
log m_0	+9.34867		cos γ	+9.13763	−9.65220
			sin d_1	+8.96508	+8.96081
(193) x_0'	+9.74646	+9.74644	cos ϕ_1 cos ϑ	−8.10271	+8.61301
y_1'	−9.23934	−9.23954	tan ϑ	1.89316	1.33812
tan N	0.50712	0.50690	ϑ	−90 43 58	+87 22 18
N	+107 16 49	+107 17 18	sin	9.99996	9.99954
sin	9.97994	9.97993		9.99591	9.95159
	9.76652	9.76651	cos	8.10680	8.66142
cos	9.47283	9.47302	cos ϕ_1	+9.99591	+9.95159
log 1 : n	+0.23349	+0.23348	cos d_1	+9.99814	+9.99818
(194) $M-N$	+46 51 5	+46 57 36	sin ϕ_1	+9.13577	−9.65038
sin ( )	+9.86390	+9.86384	tan ϕ_1	+9.13986	−9.69879
cos ( )	+9.83404	+9.83411	(199) tan ϕ	+9.14133	−9.70026
sin ψ	+9.21257	+9.21251	ϕ	+7 53 0	−26 38 0
ψ	+170 26 38	+9 22 17			
(195) cos ψ	−9.99414	+9.99414	(200) μ_1	106° 26′ 22″	157° 7′ 29
log (1)	−0.22763	+0.22762	ω	197 10 20	+69 45 11 W.
(2)	+9.41620	+9.41626		−162 49 40 E.	

(201) Middle: $8^h.7393 = 8^h\ 44^m.358$

These times may be computed by the formula $m = p$ in the same manner as shown for penumbra in Art. 57. If y_1 is used for m, then the time is when $m =$ unity; but if y is used, it is when $m = \rho$, the earth radius for the place of beginning or ending.

SECTION XII.

CURVE OF CENTRAL ECLIPSE.

108. *Formulæ.*—Under the conditions given in the preceding section, equation (188), the formulæ for outline of the shadow, Art. 81, reduce to the following simple forms (CHAUVENET, Art. 315):

$$y_1 = \frac{y}{\rho_1} \qquad\qquad y_1' = \frac{y'}{\rho_1} \tag{206}$$

$$\left.\begin{aligned} \sin\beta\,\sin\gamma &= x \\ \sin\beta\,\cos\gamma &= y_1 \end{aligned}\right\} \tag{207}$$

$$\left.\begin{aligned} c\sin C &= y_1 \\ c\cos C &= \cos\beta \end{aligned}\right\} \tag{208}$$

$$\left.\begin{aligned} \cos\varphi_1\sin\vartheta &= x \\ \cos\varphi_1\cos\vartheta &= c\cos(C + d_1) \\ \sin\varphi_1 &= c\sin(C + d_1) \end{aligned}\right\} \tag{209}$$

$$\tan\varphi = \frac{\tan\varphi_1}{\sqrt{1 - e^2}} \tag{210}$$

$$\omega = \mu_1 - \vartheta \tag{211}$$

$\vartheta =$ The Local Apparent Time.

The above formulæ are entirely rigorous, and this section is by far the most important of the whole eclipse. Compute with five-place logarithms to seconds for each ten minutes between the extreme times; and finally interpolate the latitudes and longitudes for every five minutes. But as the quantities at the ends vary rapidly, they cannot be interpolated, so that at least three intermediate 5-minute points must be computed with the others at each end of the curve.

109. The computer should first interpolate the quantities x, y, μ_1, d_1, and log sin d_1, in the Eclipse Tables for the 5-minute points. As suggested in Art. 64, if log x and log y have already been gotten for

the Rising and Setting Curve to five places of logarithms, the former can simply be copied here, and log y_1 readily gotten from y of the former curve. γ is not required in this curve, only its sine or cosine. As this curve must be closely computed, either sin β or cos β may be differenced. The difference for sin β will be large in the middle; and for cos β large at the ends, remembering that the end points are computed for a different interval. Another very good method of checking quantities in these curves, where the differences at the ends run up to infinity, is to subtract one end from the other. These differences will generally be much less than differences in the usual method; and this series can now be differenced on a slip of paper in the usual manner. The addition of $C + d_1$ may be revised if necessary.

Finally, log cos φ_1 and log sin φ_1 should be examined to see that they give the same angle within one or two units of the last decimal place of logarithms. This, as previously explained, will detect any inconsistency, but not any other kind of mistake.

In regard to differencing, it is not always necessary, but it sometimes saves much time. Small errors may be difficult to locate otherwise, especially toward the ends of the curve, or if two or three occur near together. In the present eclipse there are twenty-six computed points similar to that given in the example.

Finally, the latitudes and longitudes reduced to decimals of a minute are to be written off in a column, differenced, and then interpolated for every five minutes. In this interpolation for one or two points toward the ends fourth differences are necessary to make the interpolation difference smoothly. The coefficient for this is, with sufficient accuracy, $\frac{1}{80}$ of $\Sigma\Delta_4$, and even this will not always make the differences run smoothly.

The angle ϑ in these formulæ, as well as in all the others, when reduced from arc to time, is the *Local Apparent Time* of the phenomenon, a very useful quantity and well worth a place in the *Nautical Almanac* Tables of Eclipses. It may be reduced to Local Mean Time by applying the equation of time; and is therefore valuable to astronomers who wish to observe an eclipse, since it gives the local times without further calculation.

110. *Geometrical Illustration.*—These formulæ are essentially the same as those for Outline of the Shadow, Section VIII., after the coördinates ξ, η, and ζ have there been found. In the present section these are known, or at least readily given by formula (207);

and the geometrical explanation, therefore, does not differ from that already given in Art. 85–6, to which the reader is referred. The points of the curve on the earth's sphere are those on the centre line (Figs. 2 and 8) marked by the 10-minute points. The angle γ is that which a line drawn from any one of these points to the centre of the sphere makes with the principal meridian. In this eclipse γ varies from $-82°$ through $180°$ to $+116°$.

111. *Example.*—In the following example, besides the computation of one point of the curve of central eclipse, there is given also the work for Duration and for Northern and Southern Limits. These are all closely connected, and the two latter depending upon various quantities in the computation for the central line; for convenience of reference they are therefore given together. The two last columns will be referred to under their proper sections.

Curve of Central Eclipse, Duration, and Northern and Southern Limits of Umbra.

Example, Total Eclipse, 1904, September, $9^d\ 9^h\ 0^m$.

	Central Curve.			Duration.			N. and S. Limits.	
(207)	x	+8.98664	(219)	$\log l_1$	—8.1374	(250)	$L : \cos\beta$	—8.2718
(206–7)	y_1	—9.30325		$i \cos\beta$	+7.6536		$\log\lambda$	—1.8081
	$\tan\gamma$	9.68339		$l - l\ A$	0.4838		$\sin Q$	+9.7042
	cos	9.95457		B	0.6071	(251)	$\cos Q$	+9.9357
	$\sin\beta$	9.34868		L	—8.2607		$\sin Q \sin d_1$	+8.6668
(208)	$\cos\beta$	+9.98891	(220)	c'	+9.7429		$\tan H$	+1.2689
	$\tan C$	—9.31434		$f \cos\beta$	+9.4052		H	+86 55 6
	C	—11 39 8		$l - l\ B$	0.3375		sin	+9.9994
	cos	9.99096		A	0.0701		$\log h$	+9.9363
	$\log c$	+9.99795		a	+9.4753		$\vartheta - H$	—81 19 9
(209)	$(C + d_1)$	—6 23 16	(221)	b'	+9.2438		sin ( )	—9.9950
	sin ( )	—9.04627		$\tan Q$	0.2315		cos ( )	+9.1788
	cos ( )	+9.99730		$\sin Q$	+9.9357	(252)	$\log d\phi$	+1.7394
	$\cos\vartheta \cos\phi$	+8.99525		Numerator	—2.0537		log (1)	+9.9701
	$\tan\vartheta$	8.99139	(222)	$\log t$	—2.5784		(2)	—1.5105
	ϑ	+5 35 57		t	378.8		Nos. (1)	+0.96
	cos	9.99792		Duration	6^m 18.8		(2)	—32.40
	$\cos\phi_1$	+9.99733					$d\omega$	—31.44
	$\sin\phi_1$	—9.04422					$d\phi$	+54.88
	$\tan\phi_1$	—9.04689						
(210)	$\tan\phi$	—9.04836						
	ϕ	—6 22 40						
	μ_1	+135 41 42						
(211)	ω	+130 5 45						
		130 5.8 W.						

		Latitude.	Longitude.
(253)	N. Curve	—5° 27′.8	129° 34′.3 W.
	S. Curve	—7 17 .6	130 37 .3 W.

[For the Duration, see Art. 117; and for Limits, Art. 138.]

SECTION XIII.

CENTRAL ECLIPSE AT NOON.

112. *Formulæ.*—This caption means at *Local Apparent Noon.* It is the point, J, where the centre line of the path crosses the axis of Y (Plate I., Fig. 2, Plate II., Fig. 3), from which the criterion evidently is,

$$\vartheta = 0 \tag{212}$$

This is a point of the central curve, and by (209), $x = 0$; also by (206), (207) of the central curve

$$y_1 = \frac{y}{\rho_1} \tag{213}$$

$$\sin\beta = y_1 \tag{214}$$

By (208), $C = \beta$; and by (209–211),

$$\varphi_1 = \beta + d_1 \tag{215}$$

$$\tan\varphi = \frac{\tan\varphi_1}{\sqrt{1-e^2}} \tag{216}$$

$$\omega = \mu_1 \tag{217}$$

The Greenwich mean time of this phenomenon is the time of conjunction in right ascension, already found for the elements, Art. 21.

Equations (213–217) solve this problem, and the quantities are to be taken for the time of conjunction. If the precepts of the foregoing sections are followed, this interpolation will be for a fraction of ten minutes, and the factor in numbers from the example (Art. 21) will be 0.95686. For this time x, interpolated from the ten-minute values given in the Eclipse Tables and *Nautical Almanac,* should equal zero, and not vary from this more than a fraction of the last decimal of numbers or logarithms.

113. The following equation is devised as a check upon y_1, as it gives the value for the time of conjunction. Since $x = 0$ here, $(a - \alpha) = 0$, and equation (25) for y becomes

$$y = r\sin(\delta - d) = \frac{\sin(\delta - d)}{\sin\pi}$$

In this substitute the value of d from equation (18).

$$y = \frac{\sin\left[\delta - \delta' + \frac{b}{1-b}(\delta - \delta')\right]}{\sin\pi}$$

which is rigorously exact, but as the angles are small, write the angles for the sign and reducing,

$$y = \frac{1}{(1-b)} \frac{\delta - \delta'}{\pi} \tag{218}$$

which will give y correctly, using five-place logarithms. If $\delta - \delta'$ has been employed in the elements, as suggested, as a check on δ and δ', then all the quantities are already given. This equation will be again referred to for other explanations under Section XIX., on Prediction of Eclipses by Semidiameters, Art. 159.

114. Example, Central Eclipse at Noon.

EXAMPLE, TOTAL ECLIPSE, 1904, SEPTEMBER 9.

(218)	From the Elements (Art. 21)	$\delta - \delta'$	$-10'\ 25''.61$	-2.79631
	" " "	π	$+61\ 22\ .957$	$+3.56619$
				-9.23012
	From Main Computation (Art. 36)		$\log 1:1-b$	$+0.00103$
		y	-0.170272	-9.23115
	As interpolated as a check	y	-0.170276	
(214)	(ρ_1 from tables = 9.99854)	y_1		-9.23261
		β	$-9°\ 50'\ 14''$	
		d_1	$+5\ \ 16\ \ 2$	
(215)		$\tan(\phi_1)\ \beta + d_1$	$-4\ \ 34\ \ 12$	-8.90271
(216)		$\tan \phi \quad \phi$	$-4\ \ 35\ \ 7$	-8.90417
(217)	From Eclipse Tables (Art. 37)	$\omega = \mu_1$	$+133\ \ 5\ \ 11$ West.	

The latitude and longitude from the central line should now be interpolated for the time of conjunction, and they should agree with those just found. The interpolation is for a fraction of five minutes. The time of conjunction in the present example being (Art. 21) 8^h $49^m.57$ is $4^m.57$ after the five-minute time, and the fraction for interpolation is $+0.914$; second differences should be used.

115. *Special Cases.*—In the eclipse 1891, June 6, y at conjunction, as we see from the eclipse tables for that year, has the value about 0.9962 in numbers; log y, 9.9983, and log y_1 about 9.9945, which gives rather uncertain values for β, a fault of the formula which cannot be avoided, being near 90°. β is about 87° 20′, d_1 about + 22° 40′; consequently, $\beta + d_1 = 110°$, and $\mu_1 = 70°$. The shadow having fallen beyond the north pole, we must therefore take

$$\varphi = 180° - (\beta + d_1) = 70° \text{ north}$$
$$\omega = \mu_1 + 180° = 250° \text{ west} = 110° \text{ east.}$$

This is the eclipse noted in Art. 67 as being an *Annular Eclipse of the Midnight Sun.*

Impossible Solution.—In the eclipse of 1896, Feb. 13, y at conjunction has a value of about -1.02, y_1 will be still greater, and the equation $\sin\beta = y_1$ will show an impossible value for β. The centre line does not cross the principal meridian of the earth's sphere, and there is consequently no Eclipse at Noon. This eclipse is also referred to in Art. 67. The preceding case approaches very nearly to this: y when divided by ρ_1 is not far from unity.

The quantity y or y_1 at conjunction is the most convenient element for measuring the motion of the shadow north or south in successive eclipses of a series, since the inclination of the path changes with each eclipse. The latitude at Noon cannot be used, since the motion is rapid at the poles and slow toward the zenith of the projection. The durations cannot be used, for their differences are zero toward the equator and rapid at the poles.

SECTION XIV.

DURATION OF TOTAL OR ANNULAR ECLIPSE.

116. *Formulæ* (Chauvenet, Art. 317):

$$L = l_1 - i_1 \cos\beta \tag{219}$$

$$a = c' - f \cos\beta \tag{220}$$

$$\tan Q = \frac{a}{b'} \tag{221}$$

$$t = \frac{7200\, L \sin Q}{a} \tag{222}$$

This computation can conveniently be placed below that of the central line, on account of quantities in the former work required also here being computed for the same times as the central line. All the quantities are to be taken for the umbral cone. Compute with four-place logarithms. Regard the sign of l_1, and in fact all the signs, though t is to be taken as positive, even if the equation gives a negative value. The logarithm of b' can generally be interpolated with care, unless it is very small; in which case the natural numbers must be interpolated and the logarithms gotten from them. Log c' can always be easily interpolated here, and, as suggested in Art. 32, c' is computed for the umbral cone. Addition and subtraction logarithms will be found very convenient for these formulæ.

The above quantities are given on a basis of one hour's interval; especially b' and c', which are formed from the *hourly* motions of x and y. t would therefore result in decimals of an hour, and the constant 3600 is introduced to give t in seconds. And as the duration depends upon the *diameter* of the shadow, whereas we have used l, the radius, the factor 2 is introduced; t may result, a negative value from L being negative for total eclipse, but it must be taken as positive in all cases, since it is the numerical interval of time that is desired.

Q is not required here, only its sine. But as the cosine is required for the limiting curves, it saves time to take out both from the tables, and set down the cosine ten or twelve lines below for future use.

117. *Example.*—This will be found in Art. 111, near the computations for the Central Line. Cos β is to be taken from that computation. It need not be copied off, for i cos β and also f cos β can be written off at once, having log i and log f on a slip of paper with other constants for the eclipse. The computation need be carried no further than log t, getting t in seconds and reducing to minutes while transferring to another page, where it is differenced for errors, and then interpolated for every five minutes. One decimal of a second only is sufficient. This series of total eclipses has the longest duration of any others, except the series of 1883, 1901, etc., which is increasing, while the duration of the series of 1886, 1904 is decreasing.

118. *The Angle Q.*—By comparing equations (220), (221), of this section with equation (177) of the Limiting Curves of Penumbra, it is seen that the numerators and denominators of tan Q are transposed in the two forms, showing apparently that the two values are complements of one another. The truth is, however, that Q in this section is Q of the previous section plus 90°, which is easily shown by CHAUVENET's formulæ in vol. i., p. 493. The equations are not numbered, but are as follows, substituting L in the first members:

$$L \sin Q = \mp (x' - \xi') \frac{t}{2}$$

$$L \cos Q = \mp (y' - \eta') \frac{t}{2}$$

$$x' - \xi' = c' - f \cos \beta = a$$

$$y' - \eta' = - b'$$

In substituting the latter equations for the former, CHAUVENET omits the signs, "since it is only the numerical value of t that is required"; but if we retain the signs we can see the value of Q as follows—the upper signs are for the point of beginning of the duration, and the lower for the point of ending:

$$\left.\begin{aligned} L \sin Q &= \mp \frac{at}{2} \\ L \cos Q &= \pm \frac{b't}{2} \end{aligned}\right\} \tag{223}$$

Taking the lower signs, Q is an obtuse angle; and with the upper, an acute angle taken negatively—the values differing 180° from one another. In Fig. 15, supposing the circles to now represent the *umbral* shadow, while we plot for the northern limiting curve $Q = AMe'$, we have here for duration for the point of beginning $Q = AMm = BnM$, which is laid off in the direction of the path, the points m *and* n forming the duration. A total eclipse would reverse the conditions of the signs, on account of the negative value of L.

We read in the section on Limits that the path of the shadow on the earth's surface is shortened from the distance on the fundamental plane, and the office of Q is here to diminish the radius of the umbral cone proportionately, so that instead of the duration depending upon the radius $nM = Mm$, it depends upon the line ms, which is proportionate to Mm, as sin Q is to unity. It is as well to know these facts about the angle Q, but they are given here not as a matter of curiosity, but because this knowledge will shorten the labor of computing the umbral limiting curves by nearly one-half.

As Q in the former sections hardly, if ever, reaches 35°, and is generally much less, its value here when the signs are neglected will in effect be between + 40° and 135°.

SECTION XV.

EXTREME TIMES, NORTHERN AND SOUTHERN LIMITS OF UMBRA.

119. CHAUVENET'S *formulæ* (Art. 320):

$$\left.\begin{aligned} m \sin M &= x_0 \mp l_1 \sin E_0 \\ m \cos M &= y_0 \mp l_1 \cos E_0 \end{aligned}\right\} \tag{224}$$

$$\left.\begin{aligned} n \sin N &= x_0' \mp \frac{l_1}{e} b'' \\ n \cos N &= y_0' \mp \frac{l_1}{e} c'' \end{aligned}\right\} \tag{225}$$

The above are CHAUVENET'S formulæ, but the results will be more accurate if y_1 and y_1' be substituted for y_0 and y_0', as suggested in Art. 130 and Art. 134. Then $\gamma = \gamma'$.

$$\left\{\begin{array}{l}\text{For Total Eclipse: } - \text{for north limit, } + \text{ for south limit.}\\ \text{For Annular Eclipse reverse these conditions.}\end{array}\right\} \quad (226)$$

$$\sin \psi = m \sin (M - N) \quad (227)$$

$$\tau = \frac{\cos \psi}{n} - \frac{m \cos (M - N)}{n} \quad (228)$$

$$T = T_0 + \tau \quad (229)$$

$$\gamma = N + \psi \quad (230)$$

$$\tan \gamma' = \rho_1 \tan \gamma \quad (231)$$

$$\left.\begin{array}{r}\cos \varphi_1 \sin \vartheta = \sin \gamma' \\ \cos \varphi_1 \cos \vartheta = - \cos \gamma' \sin d_1 \\ \sin \varphi_1 = \cos \gamma' \cos d_1\end{array}\right\} \quad (232)$$

$$\tan \varphi = \frac{\tan \varphi_1}{\sqrt{1-e^2}} \quad (233)$$

$$\omega = \mu_1 - \vartheta \quad (234)$$

$$\left\{\begin{array}{l}\psi \text{ is obtuse for beginning with its cosine negative;}\\ \text{And acute for ending } \quad \text{“} \quad \text{“} \quad \text{positive.}\end{array}\right.$$

120. These formulæ are to be computed closely with five-place logarithms, because figures are given in the *Nautical Almanac*, and also because they are to be compared with the results for the central line.

It will be noticed that although the quantities x_0, y_0, E_0 are taken for the epoch hour, yet the formulæ reduce them to the time of the phenomenon. This is readily seen from CHAUVENET'S construction of the formulæ on the upper half of page 486, vol. i. It is also seen in formula (225), because we have from equation (45), Art. 32, for E,

$$\sin E = \frac{b'}{e} \qquad \cos E = \frac{c'}{e}$$

The *mean* hourly changes of x_0 and y_0 are x_0' and y_0'; they are to be taken for the times of the central eclipse. The hourly changes of b' and c' are b_0'' and c_0''; hence, the hourly changes of $l_1 \sin E_0$ are therefore $\frac{l_1}{e} b''$, and of $l_1 \cos E_0$, $\frac{l_1}{e} c_0''$; so that equation (225) is made up wholly of the hourly changes of equation (224).

The quantity $\log e$, as shown in Art. 44, has a minimum value at the middle of the eclipse, and its variation before and after this time

is symmetrical for equal times. Hence, for the times of beginning and ending of the eclipse, its values are found to be sensibly equal. That is the reason why CHAUVENET in the example (p. 487) takes e as constant and at the *ends* of the eclipse.

But one other quantity remains to be considered, the radius of umbra l_1. The formulæ take no account of its changes, and CHAUVENET in his example takes a mean value, and employs only four-place logarithms. This is not up to the standard of accuracy of the computations heretofore given where figures are required. We should take l_1 for both the times of beginning and ending of the central eclipse, giving two values of M and $\log m$; and all the data will then result for the times of beginning and ending.

121. These formulæ are the same precisely as given for the penumbra, using, however, l_1 for umbra instead of l. CHAUVENET gives very meagre directions for their use for the umbral times, and omits wholly the important fact that when used for a total eclipse the negative sign of l_1 would reverse the conditions of the signs given for penumbra. The proper signs to be taken are noted here.

On account of the several conditions imposed upon these formulæ—the number of columns, the exactness required in the use of five-place logarithms—these formulæ will probably try the computer more than any other part of the whole eclipse. The results coming out so near together, and having to agree with those of the central eclipse, small mistakes become more apparent than in the previous problems.

As the times, latitudes, and longitudes here lie close to those of the extreme times of central eclipse, the several quantities may be taken for those times without error, so that but one approximation is sufficient. It is suggested that the computer take the four columns in the order given in the example, and also mark the headings as given there with the signs to be used. Addition and subtraction logarithms are almost indispensable here. Throughout this computation the several quantities will result, so that the corresponding quantity for central eclipse will usually lie between them, the exceptions being rare. It will be noticed that the angle γ' is used in these equations, while γ is used in the central. The proportional parts for μ_1, as before noted, can be taken from Table VIII.

122. *Example.*—A part of the work is the same for the two points for beginning and for the two points for ending, so that it need not be computed for both. The double sign then gives the four

points. As suggested in Art. 22, the computer is supposed to have the constants $\log \frac{1}{e}$, $\log \frac{1}{\sqrt{1-e^2}}$, etc., written on a slip of paper, since they are not written in the example. In these equations there is some similarity of numerical values that are generally the same in all eclipses. In the portion depending upon the equations (224), (225) (see example), the values of the two northern points are generally alike, and also those for the two southern points; for example, $m \sin M$. But the angle N is alike for the two points for beginning, and a different value for the two points for ending. The numbers (1), (2) are used for the two terms of equation (228), and for the second term the two beginnings are generally similar, and the two endings. The latitudes should be checked by ascertaining if $\sin \varphi_1$ and $\cos \varphi_1$ give the same angle within at least one unit of the last place of logarithms. The differences of μ_1 for the two points of beginning and for ending can be checked by the proportional parts for the small interval of time between them.

Extreme Times N. and S. Limits of Umbra.

Example, Total Eclipse, 1904, September 9.

		Beginning.		Ending.	
		N. (−).	S. (+).	N. (−).	S. (+).
(124)	$\log l_1$	−8.13894		−8.13821	
	$\sin E_0$	+9.48029		+9.48029	
	$\cos E_0$	+9.97920		+9.97920	
(224, 226)	x_0	+8.98664		+8.98664	
	$l_1 \sin E_0$	−7.61923		−7.61850	
	$l - l$	A 1.36741	B	A 1.36814	B
		B 1.31566	A 1.34835	B 1.38656	A 1.34913
	$m \sin M$	+9.00489	+8.96758	+9.00506	+8.96769
(224)	y_0	−9.30179		−9.30179	
	$l_1 \cos E_0$	−8.11814		−8.11741	
	$l - l$	B 1.18365	A	B 1.18438	A
		A 1.15423	B 1.21121	A 1.15500	B 1.21190
	$m \cos M$	−9.27237	−9.32935	−9.27241	−9.32931
(224)	$\tan M$	9.73252	9.63823	9.73265	9.63838
	M	+151° 37′ 26″	+156° 30′ 12″	+151° 37′ 0″	+156° 29′ 46″
	cos	9.94441	9.96241	9.94938	9.96239
	$\log m$	+9.32796	+9.36694	+9.32803	+9.36692
(225)	$\log (1 : e)$	+0.2362		+0.2362	
	b''	+8.1268		+8.1268	
	c''	−7.6182		−7.6182	

(225, 226)		x_0'	+ 9.74646		+ 9.74644	
		$(l_1 : e)b''$	— 6.5019		— 6.5012	
		$l - l$	A 3.2446	B	A 3.2452	B
	$(B - A)$		B 25	A 25	B 25	A 25
		$n \sin N$	+ 9.74671	+ 9.74621	+ 9.74669	+ 9.74619
(225)		y_0'	— 9.23788		— 9.23808	
		$(l_1 : e)c''$	+ 5.9933		+ 5.9926	
		$l - l$	A 3.2346	B	A 3.2455	B
	$(B - A)$		B 24	A 24	B 24	A 24
		$n \cos N$	— 9.23812	— 9.23764	— 9.23832	— 9.23784
(225)		$\tan N$	0.50859	0.50857	0.50837	0.50835
		N	+107° 13′ 31″	+107° 13′ 33″	+107° 14′ 0″	+107° 14′ 3″
		sin	9.98007	9.98007	9.98005	9.98005
		$\log (1 : n)$	+ 0.23336	+ 0.23386	+ 0.23336	+ 0.23386
(227)		$M - N$	+44° 23′ 55″	+44° 16′ 39″	+44° 23′ 0″	41° 15′ 43″
		sin ( )	+ 9.84488	9.87960	+ 9.84476	+ 9.87950
		cos ( )	+ 9.85400	9.81451	+ 9.85411	+ 9.81465
		$\sin \psi$	+ 9.18284	9.24654	+ 9.17279	+ 9.24642
		ψ	+171° 26′ 16″	+169° 50′ 20″	+8° 33′ 40″	+10° 9′ 30″
(228)	Beginning— Ending+	$\cos \psi$	— 9.99513	— 9.99314	+ 9.99514	+ 9.99314
		log (1)	— 0.22849	— 0.22700	+ 0.22850	+ 0.22700
		(2)	+ 9.41532	+ 9.41531	+ 9.41550	+ 9.41543
	First term	Nos. (1)	— 1.69235	— 1.68654	+ 1.69238	+ 1.68654
	Second term	— (2)	— 0.26020	— 0.26020	— 0.26036	— 0.26027
		τ	— 1.95255	— 1.94674	+ 1.43202	+ 1.42627
(229)		T	7.04745	7.05326	10.43202	10.42627
			7^h 2.8470	7^h 3.1956	10^h 25.9212	10^h 25.5762
(230)		γ	+278° 39′ 47″	+277° 3′ 53″	+115° 47′ 40″	+117° 23′ 33″
		$\tan \gamma$	0.81712	0.90682	0.31579	0.28552
(231)		$\tan \gamma'$	0.81566	0.90536	0.31433	0.28406
		γ'	+278° 41′ 31″	277° 5′ 18″	+115° 52′ 12″	+117° 28′ 15″
(232)		$\sin \gamma'$	— 9.99498	— 9.99667	+ 9.95414	9.94804
		$\cos \gamma'$	+ 9.17932	+ 9.09131	— 9.63981	9.66398
		$\sin d_1$	+ 8.96508	+ 8.96508	8.96081	8.96081
		$\cos \phi_1 \sin \vartheta$	— 8.14440	— 8.05639	8.60062	8.62479
		$\tan \vartheta$	1.85058	1.94028	1.35352	1.32325
		ϑ	—90° 48′ 29″	—90° 39′ 27″	+87° 27′ 47″	+87° 16′ 49″
		sin	9.99996	9.99997	9.99957	9.99951
		$\cos \phi_1$	+ 9.99502	+ 9.99670	9.95457	9.94853
		$\cos d_1$	+ 9.99814	9.99814	9.99818	9.99818
		$\sin \phi_1$	+ 9.17746	9.08945	9.63799	9.66216
(233)		$\tan \phi_1$	+ 9.18244	9.09275	9.68342	9.71363
		$\tan \phi$	+ 9.18391	9.09422	9.68489	9.71510
		ϕ	+8° 41′ 0″	+7° 4′ 53″	—25° 49′ 25″	—27° 25′ 33″
(234)		μ_1	106° 23′ 49″	106° 29′ 3″	157° 10′ 55″	157° 5′ 44″
(234)		ω	+197° 12′ 18″	+197° 8′ 30″	+69° 43′ 8″	+69° 48′ 55″
			—162 47 42	—162 51 30		

123. The above results may now be compared with those of central eclipse in the following manner:

	Times.		ϕ.		ϑ.		ω.	
N. Limit begins	7^h $2^m.85$		$+8°$ $41'.0$		$-90°$ $48'.49$		$-162°$ $47'.7$	
		$+.17$		-48.0		$+4.51$		-2.0
Central "	7 3 .02		7 53 .0		90 43 .98		162 49 .7	
		$+.18$		-48.1		$+4.53$		-1.8
S. Limit "	7 3 .20		+7 4 .9		−90 39 .45		−162 51 .5	
S. Limit ends	10 25 .58		+27 25 .5		+87 16 .82		+69 48 .9	
		$+.11$		-47.5		$+5.48$		-3.7
Central "	10 25 .69		26 38 .0		87 22 .30		69 45 .2	
		$+.23$		-48.6		$+5.48$		-2.1
N. Limit "	10 25 .92		+25 49 .4		+87 27 .78		+69 43 .1	

124. And to show the changes during successive periods of the Saros, the *Nautical Almanacs* furnish the data:

	Durations.		At Noon.	Extreme Duration of Totality.
Eclipse.	Extreme.	Central.	ϕ.	
1868, Aug. 17,	5^h $14^m.6$	3^h $25^m.3$	$+10°$ $27'.2$	
1886, Aug. 28,	5 14 .0	3 24 .2	+ 2 58 .5	6^m $33^s.6$
1904, Sept. 9,	5 12 .9	3 22 .7	− 4 35 .1	6 23 .7

125. These quantities difference very much more smoothly than those of any other eclipse I can remember. In CHAUVENET'S example the times of the limits for beginning differ from the central times by the quantities $0^m.96$ and $0^m.54$, and the corresponding longitudes differ by 43′ and 35′. After computing these times, I found a small mistake affecting the four columns, which changes some of the results a few tenths in the decimal; but it was too late to make the change in the *Almanac*. The errata, however, appear in the *Almanac* for 1906.

These differences are generally so irregular that the comparison with the central times cannot be relied upon at all as a check upon their accuracy. I have noticed that a large eclipse like the present, where the shadow path approaches the zenith point of the projection, will difference more smoothly than one in which the centre of shadow passes near the poles.

126. It will be noticed by reference to Fig. 8, Plate V., that the line $a\,b$, Fig. 8, which makes the angle E with the meridian, and is tangent to the sphere, is *not* quite at right angles to the centre line of the path. If the inclination of the path were such that this line should be at, or nearly at, right angles to the path, then the central eclipse would begin *before* either of the limiting curves; and the central times, instead of lying *between* them, as in the comparison above given, would lie outside of them—a rather unusual occurrence; but it happened in the eclipse of 1894, Sept. 28. The longi-

tudes are published in the *Almanac*, but not the times; they both show this peculiarity, and are as follows:

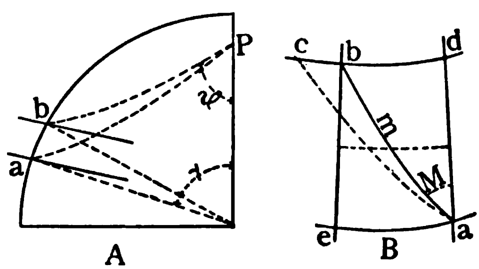
S. curve ends, 1894, Sept. 28,	$19^h\ 14^m.29$.	Longitude, $-162°\ 41'.2$
Central "	19 14 .11	$-162\ 43.5$
N. curve "	19 14 .48	$-162\ 36.5$

Analogous to this in the present eclipse, it is seen, in Art. 92, that the extreme times of central eclipse lie *outside* of the limits of the penumbra, which is also unusual.

127. *Check Formulæ.*—There is no rigorous check that I know of to these times and geographical positions. Some years ago, however, I devised a very simple formula which will check the *differences* between the quantities, though it cannot check the quantities separately.

Navigators will doubtless recognize these formulæ as a transformation of those used in *Middle Latitude Sailing*, and they are easily derived. As the shadow in the present eclipse first strikes the earth for the southern limiting curve at the point a (Fig. 16, *A*), the parallels of latitude of a and b, with their hour angles, will form a quadrilateral on the earth's surface (Fig. 16, *B*), of which ab is the diagonal.

FIG. 16.



The earth revolves a little before the first point of the northern limiting curve is formed, which is the point c of Fig. 16, *B*. In this latter figure $db = ae$ is the difference of the hour angles $\Delta\vartheta$; ad the difference of latitude, $\Delta\varphi$; ab is the arc of a great circle $= m$ of the formulæ and in the horizon of the first figure. bc is the difference between the quantities μ_1 for these two points, and dc their difference of longitude. The line $a\,c$ cannot be geometrically shown, and it is seen that the difference of hour angles, and not of longitudes, must be used in the formulæ. They are as follows:

$$\left.\begin{aligned} m \sin M &= \cos\varphi\,\Delta\vartheta \\ m \cos M &= \Delta\varphi \end{aligned}\right\} \qquad (235)$$

$$m = \Delta\gamma \qquad (236)$$

Cos φ here is the mean of the two latitudes of the points a and b. In fact, the quantities are so small it makes little difference whether the latitude of either a or b or the mean is used, or if the geocentric latitudes are substituted. As $\Delta\varphi$ and $\Delta\vartheta$ are taken in minutes, m will also result in minutes.

Four-place logarithms are amply sufficient for these formulæ. It will be noticed that while γ is used for the central curve, γ', *not* γ, is used for the limiting curves, and this value must be used in the formulæ above. Signs may be wholly neglected here, considering only numerical differences. All the quantities required are given in the computations (Articles 107 and 122). The times are checked by these formulæ only indirectly, and by inference—that if these quantities check correctly, we may assume that the times are correct.

128. Example of the Check Formulæ.

Check on the Extreme Times, Limits of Umbra.

Between Central Point and Limits.

		Beginning. N.		Beginning. S.		Ending. N.		Ending. S.	
(135) (Art. 123)	$\Delta\vartheta$	4.51	0.6542	4.53	0.6561	5.48	0.7388	5.48	0.7388
"	$\cos\phi$		9.9954		9.9963		9.9528		9.9498
	$m \sin M$		0.6496		0.6524		0.6916		0.6886
(Art. 123)	$m \cos M = \Delta\phi$	48.0	1.6812	48.1	1.6821	48.3	1.6839	47.5	1.6767
	$\tan M$		8.9684		8.9703		9.0077		9.0119
	$\cos M$		9.9981		9.9981		9.9978		9.9977
	m	48.20	1.6831	48.31	1.6840	48.54	1.6861	47.75	1.6790
(136)	$\Delta\gamma$	48.06		48.15		48.39		47.66	
Check		+ 0.14		+ 0.16		+ 0.15		+ 0.09	

Angles of Position.

N.	γ'	278° 41′.51	
Cent.	γ	277 53 .45	−48.06
S.	γ'	277 5 .30	−48.15
N.	γ'	115 52 .20	
Cent.	γ	116 40 .59	+48.39
S.	γ'	117 28 .25	+47.66

Between the Two Limiting Points.

		Beginning.		Ending.	
(135) (Art.123)	$\Delta\vartheta$	9.04	0.9562	10.96	1.0398
	$\cos\phi$		9.9959		9.9513
	$m \sin M$		0.9521		0.9911
(Art.123)	$m \cos M = \Delta\phi$	96.1	1.9827	95.8	1.9814
	$\tan M$		8.9694		9.0097
	$\cos M$		9.9981		9.9977
	m	1 36.52	1.9846	1 36.32	1.9837
(136)	$\Delta\gamma'$	1 36.21		1 36.05	
Check		+ 0.31		+ 0.27	

In the above example the formula is first applied to the extreme points of the central curve and limits, and below that to the points of the two limiting curves.

The check errors are not in the check formulæ, but in CHAUVENET'S approximations; because the former, while theoretically approximate, are nearly rigorous as applied to such small quantities as here used. And, moreover, the following rigorous formulæ give results agreeing very closely with the results of the check formulæ:

$$\cos m = \sin \varphi_1 \sin \varphi_2 + \cos \varphi_1 \cos \varphi_2 \cos (\varDelta \vartheta) \tag{237}$$

$$m = \frac{\sin m \sqrt[3]{\sec m}}{\sin 1'} \tag{238}$$

The first of these requires seven-place logarithms, and is entirely rigorous, φ_1 and φ_2 being the latitudes of the two points previously used; but the formula has the disadvantage of giving a small angle by its cosine. If, however, the angle is uncertain, the second formula may then be used, employing only four- or five-place logarithms; the $\sin m$ is to be found from the cosine of the first formula and dividing by $\sin 1'$; m results in minutes and the formula is accurate to a small fraction of a *second.*

129. *Errors in* CHAUVENET'S *Formulæ.*—It is to be regretted that Professor CHAUVENET passed over this subject so briefly in a single page, merely referring to his previous formulæ for the penumbra which are to be used here. By his using only four-place logarithms in the example when the formulæ will bear five or more, and taking l_1, an approximate value for the middle of the eclipse, we are led to question whether he was entirely satisfied with the formulæ.

That the reader may see at a glance what errors in the formulæ are suspected, the following comparisons of the times of ending are given for the eclipse of 1903, Sept. 20. This eclipse is selected because the variations are large and the geographical positions are in high latitudes.

TOTAL ECLIPSE, SEPTEMBER 20, 1903.

CENTRAL ECLIPSE AND LIMITS OF UMBRA AT ENDING.

From the *Nautical Almanac:*

	ϕ.		ω.	
N. Limit ends	—80° 48′.2		—179° 8′.2	
		—1 121		+35.9
Central "	82 0 .3		178 32 .3	
		—0 271		+21.6
S. Limit "	—82 27 .4		—178 10 .7	

From the computing sheets:

			ϑ.		$\gamma\ \gamma'$.	
N. Limit ends	$17^h\ 27^m.84$		+82° 43′.20		γ' 170° 50′.92	
		—2.11		—1 7.53		+1 13.00
Central "	17 25 .73		81 35 .67		γ 172 3 .92	
		—0.61		—030.67		+0 27.45
S. Limit "	17 25 .12		+81 5 .00		γ' 172 31 .37	

These quantities have been checked by the formula in Art. 127, and are correct within $0'.2$, so that there is no error of the computation to be considered. The irregularities before mentioned are seen in all the quantities, especially in the latitudes. The centre line being in the middle between the two limits, it is not reasonable to suppose there should be such unequal differences as shown above.

An examination for these irregularities leads to the discovery also of two other sources of error in the formulæ, for all of which the corrections are quickly and easily made.

130. *First.*—An error in applying the formulæ; that is, in taking no account of the change in E during the intervals between the beginnings or endings of the northern and southern curves, when the error is sufficiently great to have any appreciable effect. This is well illustrated in the annular eclipse of July 18, 1898, from which we have as follows:

Change of E for one hour, $6^\circ\ 16'.9$, or for one minute,	$6'.28$
Interval between times of beginning of N. and S. limits,	$6^m.7$
Change of E in this interval,	$42'\ 5''$

This, however, is rather an extreme case. No account of any such change is provided for in the formulæ, because, first, the times of limits are unknown, and consequently their interval from the central; and, secondly, because the quantities x', y'', l_1 are taken for the times of central eclipse, and the computation made for those times. The above is the total change between the limits; in practice it would be computed for each of the four points and applied to E_0 in the formula (224). The correction is large only during the summer and winter solstices, when $\sin d$ is numerically large; and also when the path is much inclined to the axis of X, both of which augment the quantity b', equation (40). The intervals of time are large when the centre line strikes the earth very obliquely. This correction would require two approximations of the times.

131. *The second error* is small, affecting the beginnings alike and the endings alike, but with contrary signs. In CHAUVENET'S approximation of his equation (530), page 480, vol. i., or equation (174) of the present work, he places the small quantities equal to 0; and this constitutes the second error. If they are retained, we have

$$\sin(Q - E) = \frac{-a' + \zeta f \sin(Q - F)}{e} \qquad (239)$$

When this is used for limiting curves, ζ has a value approaching unity, and the second term becomes more correct toward the middle of the eclipse and a' becomes equal to 0, so that CHAUVENET'S approximations are correct, especially as it is shown in Art. 95 that the error will be along the curve, and not laterally.

But when this formula is applied to the extreme times, it is not a *line*, but a *point*, which is required; a' is a maximum, and the error becomes apparent. For these times, the points being in the fundamental plane, $\zeta = 0$, and we have rigorously

$$\sin(Q - E) = -\frac{a'}{e} \tag{240}$$

For a', equation (48)

$$a' = -l' - \mu_1' i x \cos d \tag{241}$$

This last formula shows that the correction varies with x; it is 0 in the middle and has contrary signs at the ends.

The above corrections for the total eclipse, Sept. 9, 1904. $x =$ unity, or very nearly, at the beginning and end.

	Beginning.	Ending.
$\mu_1 i_1 x \cos d$	$-$ 0.001205	$+$ 0.001205
l_1'	$+$ 51	$-$ 37
$\log(-a')$	$-$ 7.06221	$+$ 7.06744
$\log 1 : e$	$+$ 0.23620	$+$ 0.23617
$\log \sin(Q - E) =$	$-$ 7.29841	$+$ 7.24361
$Q - E =$	$-$ 0° 6′ 50″	$+$ 0° 6′ 55″

This is easily computed, for the quantities are all given in the computation, and E_0 for the epoch hour corrected by these results will give their effect to the beginning and ending.

This correction varies with x more than with any other factor, so that it may be near its maximum for this eclipse.

132. *The third error* is the most important of all, for it is perhaps the whole source of the irregularity noticed in the tabulated quantities of Art. 129. The error is that y_1 and its hourly motion y_1' should be used in the formulæ instead of y and y'. CHAUVENET'S formulæ give the times, using y and y' *without taking any account of the compression of the earth.* He then considers the compression by reducing γ to γ' for the latitudes and longitudes; but this is not enough: the compression affects the times and the quantities M, m, N, n, and through these ψ, and finally γ. The times affect μ_1 and the longitude.

CHAUVENET, in discussing the extreme times of the limits of

penumbra, had occasion to refer to his equation (514), which is $\xi^2 + \eta_1^2 = 1$, in which he substitutes η, and naturally uses y instead of y_1 in the equations following. The times are unimportant, and used merely as a check on positions computed only for the chart in the *Nautical Almanac*, so that the error is not perceived here and the equations are sufficiently correct for the penumbra. But when he gives the formulæ for the umbral cone, he briefly refers to these for penumbra with the remark that l_1 for umbra should be used instead of l.

The proper quantities to be used here are y_1 and y_1'. They are used for the extreme times of central, the central curve, outline from which all the other formulæ are in a measure derived. These quantities are *not* used for the unimportant problems of rising and setting curves, maximum, the limiting times and curves of penumbra, and finally for the extreme times generally. In this latter case he has made use of a simpler device than employing ρ_1 to take account of the compression. He finds the radius of the earth, p, for the point of first and last contacts, and this quantity also enters into the formulæ for ψ and γ.

133. To show the effect of using y_1 and y_1' upon the quantities given in Art. 129, these have been recomputed, with the following results; the quantities on the lower lines can be compared with those previously given:

Comparison of Extreme Times, Limits of Umbra, at Ending.

Total Eclipse, 1903, September 20.

	Using y and y'.		Using y_1 and y_1'.	
	N. Curve.	S. Curve.	N. Curve.	S. Curve.
M	+185° 47′ 40″	185° 57′ 47″	185° 46′ 29″	185° 56′ 35″
$\log m$	9.95941	9.96539	9.96087	9.96684
N	107° 51′ 55″	107° 52′ 22″	107° 55′ 21″	107° 55′ 56″
$\log 1 : n$	0.24527	0.24542	0.24515	0.24528
ψ	62° 57′ 10″	64° 37′ 30″	63° 18′ 0″	64° 59′ 40″

Also the following results, using y_1' and y_1':

	ϕ.		ω.	
N. Limit ends	−81° 10′.4	−49′.9	−179° 0′.6	+28′.3
Central "	82 0 .3	−50 .9	178 32 .3	+39 .7
S. Limit "	−82 51 .2		−177 52 .6	

			ϑ.		γ.	
N. Limit ends	17h 27m.11	−1m.38	+82° 24′.42	−0 48.75	−171° 13′.35	−50′.57
Central "	17 25 .73	−1 .42	81 35 .67	1 1.10	172 3 .92	−51 .68
S. Limit "	17 24 .31		+80 34 .57		−172 55 .60	

Quantities from the central curve are unchanged. The above results are checked by the formulæ of Art. 127 and are correct within 0′.2. They seem to be much more in accordance with what we might reasonably expect than those in Art. 129 given by CHAUVENET'S formulæ. The differences between the values for central eclipse and the two limits, it seems, should be nearly if not exactly equal.

134. *Corrected Formulæ.*—For the convenience of the computer the changes above suggested in the formulæ of this section are as follows:

Art. 131, $$a' = -l_1' - \mu_1 i_1 x \cos d \qquad (242)$$

Art. 131, $$\sin(Q_0 - E_1) = -\frac{a'}{e} = \frac{l_1 - \mu_1 i_1 x \cos d}{e} \qquad (243)$$

Art. 130, $$Q = Q_0 + t\Delta E_1 \qquad (244)$$

Art. 132, $$y_1 = \frac{y}{\rho_1} \qquad y_1' = \frac{y'}{\rho_1} \qquad (245)$$

$$\left.\begin{aligned} m \sin M &= x_0 \mp l_1 \sin Q \\ m \cos M &= y_1 \mp l_1 \cos Q \end{aligned}\right\} \qquad (246)$$

$$\left.\begin{aligned} n \sin N &= x_0' \mp \frac{l_1}{e} b'' \\ n \cos N &= y_1' \mp \frac{l_1}{e} c'' \end{aligned}\right\} \qquad (247)$$

t is the interval of time of the limits *after* the central.

ΔE is the change of E in one unit of the time t.

The rest of the formulæ of Art. 119 remains unchanged *except* that γ needs no reduction to γ'.

It will be noticed that on account of the corrections in equation (243), it is no longer E that is used in these formulæ, but Q. The correction, equation (244), requires two approximations of the times, the first being used to obtain the interval t.

SECTION XVI.

CURVE, NORTHERN AND SOUTHERN LIMITS OF UMBRA.

135. *Remarks.*—In this curve ζ or $\cos\beta$ is not known, but as the curve lies very near central line, $\cos\beta$ is taken from that curve, and the formulæ are, therefore, not rigorously exact, but are sufficiently so for ordinary purposes, and the error becomes less as $\cos\beta$ becomes greater—that is, toward the middle of the eclipse where greater accuracy is desired.

As ζ or $\cos\beta$ enters as a divisor in the equation for λ, near the ends where it is small λ becomes very large and at the extreme points becomes infinite. Nevertheless, I have computed points of this curve one minute after the commencement of the central curve with good results for the limiting curves. These errors of the formulæ are of little account, for no important observations are made at the ends of the eclipse, because the sun is low in the horizon and much affected by errors of refraction and parallax.

136. *Formulæ* (Chauvenet, Art. 320):

For Q.
$$\tan\nu = \frac{f}{e}\cos\beta \tag{248}$$

$$\tan(Q - \tfrac{1}{2}E) = \tan(45° + \nu)\tan\tfrac{1}{2}E \tag{249}$$

Curve
$$\begin{aligned}\lambda &= \left(\frac{l_1}{\cos\beta} - i_1\right.\frac{1}{\sin 1'} \\ &= \frac{L}{\cos\beta\sin 1'}\end{aligned} \tag{250}$$

$$\left.\begin{aligned}h\sin H &= \cos Q \\ h\cos H &= \sin Q\sin d_1\end{aligned}\right\} \tag{251}$$

$$\left.\begin{aligned}d\varphi &= \lambda h\sin(\vartheta - H) \\ d\omega &= \lambda h\cos(\vartheta - H)\tan\varphi_1 + \lambda\sin Q\cos d_1\end{aligned}\right\} \tag{252}$$

$$\left.\begin{array}{lll}\text{For Total Eclipse North Curve is} & \varphi + d\varphi & \omega + d\omega \\ \text{South Curve is} & \varphi - d\varphi & \omega - d\omega \\ \text{For Annular Eclipse reverse these conditions.} & & \end{array}\right\} \tag{253}$$

137. *The Angle Q.*—The formulæ for Q are the same as those used for the penumbral curves (Arts. 93 and 98). And by comparing the formula (177) for limits of penumbra, with (221) for duration of central eclipse, it is seen that Q for limits of penumbra is the complement of Q for duration; consequently, the value of Q need not be computed for this curve if the duration has already been computed. For tan Q is there given, from which its sine is used in that curve; and it is suggested at the end of Art. 116 that the cosine be also taken out of the tables at the same time and written ten or twelve lines below for use in this computation. It should here be marked *sine;* and the sine in duration can now be copied from the work, placed below the *sine* here and marked *cosine.* This suggestion reduces the computation of this curve nearly one-half.

An example of the computation of Q is here given for the eclipse of Sept. 9, 1904. The cosine agrees exactly with that used in the duration, Art. 111, but the sine differs two units of the last decimal.

To make them agree exactly, either one or both would have to be carried out to seconds, or at least to tenths of a minute of arc.

For these formulæ two values of Q are given, differing 180°; but as the numerical values depending upon them are the same with different signs, only the acute value of Q is to be taken, and the change of signs is taken account of in equation (253), above given.

EXAMPLE, COMPUTATION FOR Q, LIMITS OF UMBRA.

			(249) E	$+17° 36'$
	$T.$	$9^h. 0^m.$	$(1:2)\,E$	8 48
From Central Curve	$\cos\beta$	$+9.9889$	tan	9.1898
From Eclipse Table	f	$+9.4163$	$\tan[Q-(1:2)E]$	9.5981
" " "	$\log(1:e)$	$+0.2365$	$Q-(1:2)E$	$+21$ 37
(248)	$\tan\nu'$	$+9.6417$	Q	$+30$ 25
	ν'	$+23$ 40	$\sin Q$	$+9.7044$
	$\tan(45°+\nu')$	$+0.4083$	$\cos Q$	$+9.9357$

138. *Example.*—This will be found in Art. 111 with that of the central curve. Compute with four-place logarithms to seconds or a fraction of a minute for the same times as the central curve; and this computation can conveniently be placed below that for duration, since quantities from both of these computations are required here. $E \log e\, l_1$, etc., must be taken as variable for the times heading the columns for central curve. Cos β tan φ_1, ϑ are to be taken from the central curve, l_1 or L from the duration; also sin Q and cos Q as above noted, or else computed for this curve. The signs for these are the same as those for E.

For λ the first form of equation (250) is CHAUVENET'S, but the second form will be found more convenient, taking L from duration, having regard to its sign, and cos β from central without copying either; then with $1 \div \log \sin 1'$ (3.5363), λ is at once gotten and may be conveniently placed above sin Q. Sin d_1 is interpolated for 10 minutes in the Eclipse Tables, and need not be copied here, but used directly from the tables. H is an angle always near $+90°$ and varying slowly. $\vartheta - H$ varies between the ends of this curve about 160° from near 180° through 270° to near 0°.

$d\varphi$ and $d\omega$ need not be written here as in the example, but carried over to another page, placed in columns, and differenced for errors, and interpolated for every five minutes. They can then be applied to the latitudes and longitudes of the central line, giving its limiting curves.

These quantities vary so rapidly at the ends that five-minute points must be computed as directed for the central curve, and even then for

the succeeding one or two intermediate points fourth differences must be used, for which the coefficient is $+\frac{1}{90}\Sigma\Delta_4$.

139. *Peculiarities of these Curves.*—In some eclipses the values of $d\varphi$ and $d\omega$ become very large at the ends. In the total eclipse of Aug. 8, 1896, $d\omega$ is over 5°, and the line joining the northern and southern limiting points makes a very acute angle with the central line. And generally the line across the path whose extremities generate the limiting curves is far from being a normal to the path. Moreover, $d\varphi$ sometimes changes the sign for one or two points at the very ends. This is also the case in the above-mentioned eclipse, in which the peculiarity is seen (Plate IV., Fig. 7, *H*) that the latitude of the southern point is *greater* than that of the corresponding northern point of the limiting curves. As published in the *Nautical Almanac,* the latitudes and longitudes stand thus:

G. M. T.	Northern Limit.		Central Line.		Southern Limit.	
Limits.	+63° 10′.6	0° 57′.4 W.	+62° 51′.5	0° 20′.5 W.	+62° 18′.2	0° 49′.2 E.
15^h 55	67 20 .0	12 6 .9 E.	67 46 .1	17 23 .6 E.	68 12 .2	21 46 .3
16 0	71 24 .3	34 53 .6	71 3 .8	38 31 .8	70 43 .3	42 10 .0
16 5	+72 43 .4	52 16 .0	+72 6 .2	54 55 .9	+71 29 .0	57 35 .8

This eclipse is reproduced from the chart of the *Nautical Almanac,* in Fig. 18, Plate VII., wherein this peculiarity is seen to be correct. As this eclipse is of the same series as CHAUVENET'S example, the same peculiarity is seen on his page (503), the latitude of the southern point of the limiting curve at 1^h 0^m is 50° 57′, and greater than that of the northern curve, which is given 50′ 18. This looks very much like a mistake, but it is correct.

At the beginning and end of the eclipse the earth's surface is inclined at a great angle, nearly 90° to the fundamental plane, and consequently at a very small angle to the cone of total shadow, so that for a very small change of the angle Q round the circle of shadow, the points will be moved, one west, the other east, at great distances geographically on the earth's surface. This is the explanation of the increased values of these quantities at the ends of the eclipse.

140. Another peculiarity sometimes seen the computer may meet with: For the point nearest to the extreme points there will sometimes be a point to one curve and none to the other. In the eclipse of Feb. 22, 1887, the north curve begins at 7^h $55^m.9$, central at 7^h

$58^m.7$, south curve at $8^h\ 1^m.1$. A point of the central curve is computed at $8^h\ 0^m.0$, with the durations and limits, but as the northern extreme point begins *before* this time, there can be no point of the curve, for it has not yet commenced. The geographical position can of course be plotted, but it will be found to lie on the *other side* of the maximum curve. At the end of this eclipse the same thing occurs: the south curve has ended before the central, and there is no point to the north curve. For these points a blank is left in the *Almanac* in the column of geographical positions. I remember having handed in several eclipse pages showing this peculiarity, but the line has evidently been struck out before printing, and the peculiarity thereby lost.

This completes the computations for the eclipse. The chart, however, is yet to be made and the geographical positions plotted thereon, for which the reader is referred to the next section.

SECTION XVII.

THE CHART.

141. THE curves and geographical positions computed by the foregoing formulæ are now to be plotted upon a chart, showing the whole eclipse at a glance. Generally, and except in unusual cases, the eclipse will resemble either one of the two charts here reproduced: Fig. 17, Plate VI., from the original drawings made by the author for the *Nautical Almanac*, and Fig. 18, Plate VII., from the chart in the *Nautical Almanac*, as the original drawing could not be found. The former is the total eclipse of 1904, Sept. 9, taken as an example throughout this work. It shows the two ovals forming detached branches of the rising and setting curves, connected by *two* limiting curves of the penumbra.

The second figure is the total eclipse of 1896, Aug. 8, of the same series as that of CHAUVENET'S example, the eclipse of July 19, 1860, and selected for that reason. In this figure the rising and setting curve is continuous—in shape, that of a distorted figure 8, the ends connected by *one* limiting curve of penumbra. This is the most common form of eclipse, the previous form occurring only in very large eclipses.

In partial eclipses the shape is like Fig. 18, but the central line or path is wanting. They dwindle down to special forms mentioned in Art. 67 in very small eclipses.

142. *Kinds of Projection.*—There are three which may be available, the *Globular, Equidistant,* and *Stereographic.*

143. *The Globular Projection.*—All points of the sphere are projected obliquely by converging lines, piercing the primitive circle and meeting in a point in the axis of that circle, at a distance below the lower surface of the sphere equal to the square root of 2. The condition is, that points of the sphere of 45° elevation shall be projected exactly half-way between the primitive circle and its centre, and the converging point results as above. This projection distorts the surface of the sphere toward the borders of the drawing.

144. *The equidistant projection* is a modification of the above. When the pole is the primitive circle, the semidiameters representing the equator and principal meridian are divided into nine equal parts; also the four quadrants are likewise so divided, and circles are drawn through these points, three of which determine the circle. These form the meridians and parallels of latitude. When the pole is elevated, a slight modification of this appears necessary. This, like the former projection, distorts the surface of the earth as the circles do not all meet at right angles, especially near the corners of the drawing. This is probably the projection made use of by the English *Nautical Almanac.*

145. *The Stereographic Projection.*—This is also made in the fundamental plane by converging lines which meet in the axis of the primitive circle and at the lower surface of the sphere.* This projection contains a number of important properties, the two principal of which are that all circles of the sphere project into circles, and also that the parallels of latitude and meridians intersect one another at right angles. On this account there is but little distortion; but parts are magnified toward the borders of the drawing.

* Professor CHARLES DAVIES, in several of his excellent works on geometry and projections, is mistaken in saying that the eye is supposed to be at this point on the lower surface of the sphere; for the eye in this projection has *no geometrical position* as it has in perspective, isometric drawings, etc. This point is merely a point of convergence for all projecting lines. The eye simply views the drawing from above.

In this projection it is possible to project more than a hemisphere, an example of which may be seen in the Transit of Venus Charts in the *Nautical Almanac* for 1882.

This projection is better by far than either of the two others. It can generally be made in a graphic manner by the principles of descriptive geometry; but when the author took up the subject of eclipses for the *Nautical Almanac,* he found this method to be impracticable for drawings of the size required. He, therefore, devised a series of formulæ and tables by which the projections can readily be made. The method was subsequently published in a pamphlet entitled, *Treatise on the Projection of the Sphere,* which includes both Stereographic and Orthographic Projections.

146. *The Drawing.*—This and the kind of projection to be selected may be determined by the purposes for which the drawing is intended, or may be left to the judgment of the computer. When the author took up the computation of the eclipses, he adopted the radius of the sphere sixteen inches, and has made the drawings on paper 22×28 inches. For large eclipses the scale must be reduced according to the size of the eclipse; and in the reduced scale the primitive circle usually shows in the drawing, as seen in the chart, Fig. 17, which is reduced five-sixths—nearly the greatest reduction that is necessary. In Fig. 17, Plate VI., the various geographical positions computed in the examples on the foregoing pages are marked by small circles.

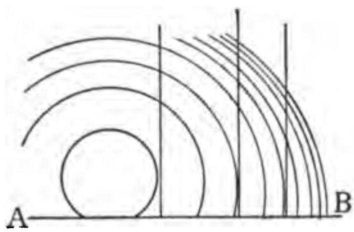
For a study eclipse or example I recommend a chart of large scale rather than small, at least as large as one-half of that adopted above.

For plotting the positions a continually varying scale is necessary, and the most convenient is a piece of *thin* paper or tracing cloth, about six inches long, and three at one end and half an inch at the other, ruled with converging lines into ten equal spaces for degrees. This can be folded to fit between the ten-degree meridians and parallels, giving degrees, the minutes to be estimated.

The charts are plotted upon the nearest ten-degree meridian to that of the *Central Eclipse at Noon,* and upon the parallel of latitude which comes nearest to the middle of the drawing, giving preference to a parallel which comes nearest to the central line in the above assumed meridian; so as to bring the central line near the assumed parallel of latitude.

147. One peculiarity of the outline curves is sometimes noticed in plotting them; at one end they appear to change their direction and curve outward. The only explanation I can give, but which does not seem to suit all cases, is that on the earth's surface they really do reverse their curvature. It is more apparent when the pole is elevated, and is very marked in the eclipse of Sept. 28, 1894. In Fig. 19, looking at the earth toward the north pole, which is elevated, A, B is the fundamental plane, and the vertical lines represent elements of several cones of penumbral shadow. It is seen that the lines on the earth *approach* the pole (their latitudes *increasing*), until they become tangent to some circle of latitude, when they change their curvature and recede from the pole.

FIG. 19.



148. For important total eclipses where the central path passes over the United States, special charts on a large size have been prepared and published in a joint report by the Naval Observatory and *Nautical Almanac* office. The two last of such publications were for the total eclipses of July 29, 1878, and May 28, 1900.

SECTION XVIII.

PREDICTION FOR A GIVEN PLACE.

149. *Preliminary Reductions.*—This section is the work of the astronomer who has located himself on or near the centre line to observe the eclipse. The latitude and longitude of the station must be accurately determined, and the latitude reduced by either of the following formulæ:

$$\left.\begin{aligned} \sin\psi &= e\sin\varphi \\ \rho\sin\varphi' &= (1-e^2)\sin\varphi\sec\psi \\ \rho\cos\varphi' &= \cos\varphi\sec\psi \end{aligned}\right\} \qquad (254)$$

From the first of these, $\cos\psi = \sqrt{1-e^2\sin^2\varphi}$ (255)

By reducing and introducing the following auxiliaries:

$$\left.\begin{aligned} F &= \frac{1}{\sqrt{1 - e^2 \sin^2 \varphi}} \\ G &= \frac{\sqrt{1 - e^2 \sin^2 \varphi}}{1 - e^2} \end{aligned}\right\} \tag{256}$$

We have
$$\left.\begin{aligned} \rho \sin \varphi' &= \frac{\sin \varphi}{G} \\ \rho \cos \varphi' &= F \cos \varphi \end{aligned}\right\} \tag{257}$$

Equation (254) gives the reduction to the geocentric latitude, but as the two factors in equation (257) are not needed separately, the latter is the most convenient. F and G are tabulated in Table XII., which is taken from the *Nautical Almanac*.

Next, $\log \mu_1'$ must be reduced to parts of radius.

We have $\Delta\mu_1$, the change of μ_1 in 1 hour in *seconds* of arc,

$\dfrac{\Delta\mu_1}{3600}$, the change of μ_1 in 1 *minute* in *minutes* of arc for N. A.,

and for the prediction in parts of radius we must have the last quantity in parts of radius; that is,

$$\begin{aligned} \mu' &= \frac{\Delta\mu_1}{3600} \sin 1' \\ &= \mu' \text{ of the Besselian Tables} \times \sin 1' \end{aligned} \tag{258}$$

When using addition and subtraction logarithms, the angle d will be needed in equation (261), which can be gotten from the sine; but when using natural numbers in equation (274), $\cos d$ is needed, which is given among the Besselian elements.

μ_1' can be reduced from the Besselian Tables in the *Nautical Almanac* by the second form of equation (258), but it will be given only to four decimals. By the first form it can easily be gotten to five decimals from the change of μ_1 given in above-mentioned table.

In the examples of eclipses given at the end of the *Nautical Almanac*, and which have been prepared by various persons, this quantity, through some oversight, has been regarded as a "constant log 7.63992." The error commenced in 1886 and has been continued to the present time. It changes with the season, and the variation is small.

The quantities $\log x'$, $\log y'$, as well also as $\log \mu'$, are given in the *Nautical Almanac* to only four decimals. It would be better if they were given to five for use with addition and subtraction logarithms.

In the examples of this computation at the end of the *Nautical Almanac*, natural numbers are used; and these logarithms will give the numbers sufficiently close, so that the succeeding terms can be given correctly to five decimals of logarithms. Log x' and log y' are simply the variations of x and y for one minute, and they can be gotten at once to five places of logarithms from the differences of x and y for ten minutes by taking the logarithms of one-tenth of these differences.

Five-place logarithms in this computation are not as important as might be supposed, for the resulting quantities are generally small, so that four figures only are generally needed in the final results, which four-place logarithms will give with care.

Each of the three assumed times becomes T_0 for the column over which it stands.

T_0 in *this* computation can be assumed at pleasure, because x' and y' are the *absolute* variations of x and y, as explained in Art. 30.

In the example in the *Nautical Almanac*, the longitude ω is called λ to conform to the notation used in the *Use of the Tables* at the end of the book.

150. *General Formulæ* (CHAUVENET, Arts. 322–24).—

$$\vartheta = \mu_1 - \omega \tag{259}$$

$$\left.\begin{aligned} A \sin B &= \rho \sin \varphi' \\ A \cos B &= \rho \cos \varphi' \cos \vartheta \end{aligned}\right\} \tag{260}$$

$$\left.\begin{aligned} \xi &= \rho \cos \varphi' \sin \vartheta \\ \eta &= A \sin (B - d) \\ \zeta &= A \cos (B - d) \end{aligned}\right\} \tag{261}$$

$$\left.\begin{aligned} m \sin M &= x - \xi \\ m \cos M &= y - \eta \end{aligned}\right\} \tag{262}$$

$$\log \mu' = \log [\mu' \text{ of the BESSELIAN Elements} \times \sin 1'] \tag{263}$$

$$\left.\begin{aligned} \xi' &= \mu' A \cos B \\ \eta' &= \mu' \xi \sin d \\ \zeta' & \quad \text{is not needed.} \end{aligned}\right\} \tag{264}$$

$$\left.\begin{aligned} n \sin N &= x' - \xi' \\ n \cos N &= y' - \eta' \end{aligned}\right\} \tag{265}$$

$$i = \tan f \quad [\text{Log tan, Angles of Cones.}] \tag{266}$$

$$L = l - i\zeta \tag{267}$$

$$\sin \psi = \pm \frac{m \sin (M - N)}{L} \tag{268}$$

$$\tau = \pm \frac{L \cos \psi}{n} - \frac{m \cos (M - N)}{n} \tag{269}$$

$$T = T_0 + \tau \quad \text{[Greenwich Mean Time.]} \tag{270}$$

For an Annular Eclipse and for Penumbra, use the upper sign.
For Umbral Cone of Total Eclipse, use the lower sign. } (271)

Local Mean Astronomical Time $T_1 = T - \omega$ (272)

Local Civil Time

If T_1 is less than 12^h write P. M. after it, and retain the date.
If T_1 is greater than 12^h subtract 12^h from it, mark the result A. M. and add 1 to the days.
This latter must not be confounded with *Standard Time.* } (273)

If natural numbers are to be used, instead of equations (260) and (261), substitute the following:

$$\left.\begin{aligned} \xi &= \rho \cos \varphi' \sin \vartheta \\ \eta &= \rho \sin \varphi' \cos d - \rho \cos \varphi' \sin d \cos \vartheta \\ \zeta &= \rho \sin \varphi' \sin d + \rho \cos \varphi' \cos d \cos \vartheta \end{aligned}\right\} \tag{274}$$

151. *Example.*—First mark the position of the given plan on the chart in the *Nautical Almanac*, and ascertain by the dotted lines the times of *beginning* and *ending* as near as possible. The *middle* time can be ascertained by comparing the longitude of the central line and limits as given to every five minutes on another page. Each of these will be T_0 for the computation following. For these times take out and interpolate the several quantities given in the tables of Besselian Elements.

Compute with four-place logarithms carefully to seconds or a

Example, Prediction for a Given Place.

Total Eclipse, 1904, September 9.

Preliminary Reductions.

Assumed position of the Given Place $\phi = -11° 54' 0''$ $\omega = 120° 0' 0''$ W.
Reduced latitude $\log \rho \sin \phi' = -9.31141$ $\log \rho \cos \phi' + 9.99062$.

From the Eclipse Chart and Longitude of the centre line.

Beginning $8^h 2^m$. Middle $9^h 31^m$. Ending $10^h 52^m$.

For these times from the Besselian Elements in the *Nautical Almanac.*

x	-0.44223	$+0.38516$	$+1.13807$
y	-0.03315	-0.28973	-0.52331
$l\ l_1$	$+0.53258$	-0.01373	$+0.53255$
μ_1	121° 11.40	143° 26.85	163° 42.25

$\Delta\mu$ from the table $15° 0' 18'' = 54018''$, whence $\log \mu' = 7.63996$

The other quantities in the logarithms vary slowly; they are given in the example where they are needed.

Computation for the Times.

	T_0	$8^h\ 2^m$	$9^h\ 31^m$	$10^h\ 52^m$
(259)	N. A. μ_1	121 11 24	143 26 51	163 42 15
	ω	120 0 0		
	$\vartheta = \mu_1 - \omega$	+ 1 11 24	+ 23 26 51	+ 43 42 15
	$\sin \vartheta$	+ 8.31739	9.59979	+ 9.83943
	$\cos \vartheta$	+ 9.99991	9.96257	+ 9.85909
(260)	$A \sin B = \rho \sin \phi'$	− 9.31141		
	$\rho \cos \phi'$	+ 9.99062		
	$A \cos B = \rho \cos \phi' \cos \vartheta$	+ 9.99053	+ 9.95319	+ 9.84971
	$\tan B$	9.32088	9.35822	9.46170
	B	− 11 49 28	− 12 51 7	− 16 8 51
	$\cos B$	9.99069	9.98898	9.98252
	$\log A$	+ 9.99984	9.96421	+ 9.86719
(261)	N. A. from $\sin d$ d	+ 5 15 42	5 14 20	+ 5 13 6
	$B - d$	− 17 5 10	− 18 5 27	− 21 21 57
	$\sin (B - d)$	− 9.46807	− 9.49210	− 9.56148
	$\cos (B - d)$	+ 9.98040	+ 9.97798	+ 9.96907
(261)	$\zeta = \rho \cos \phi' \sin \vartheta$	+ 8.30801	+ 9.59041	+ 9.83005
	N. A. x	− 9.64565	+ 9.58564	+ 0.05617
	$l - l$	A 1.33764	B 0.00477	B 0.22612
	A B	1.35716	8.04300	9.83451
	$m \sin M = x - \xi$	− 9.66517	− 7.62864	+ 9.66456
(261)	N. A. y	− 8.52048	− 9.46199	− 9.71876
	$\eta = A \sin (B - d)$	− 9.46791	− 9.45631	− 9.42867
	$l - l$ B	0.94743	0.00568	0.29009
	A	0.89542	8.11950	9.97784
	$m \cos M = y - \eta$	+ 9.41590	− 7.57581	− 9.40651
(262)	$\tan M$	0.24927	0.05283	0.25805
	M	− 60 36 29	− 131 31 25	+ 118 53 58
	$\sin M$	9.94016	9.87429	9.94224
	$\log m$	+ 9.72501	7.75435	+ 9.72232
(263)	N. A. $\log \mu_1$	+ 7.63996		
(264)	N. A. x'	+ 7.9683	+ 7.9683	+ 7.9682
	$\xi' = \mu' A \cos B$	+ 7.6305	+ 7.5931	+ 7.4897
	$l - l$ B	0.3378	0.3752	0.4785
	A	0.0707	0.1375	0.3031
	$n \sin N = x' - \xi'$	+ 7.7012	+ 7.7306	+ 7.7928
(264)	N. A. y'	− 7.4597	− 7.4599	− 7.4600
	N. A. $\sin d$	+ 8.96238	+ 8.96050	+ 8.95880
	$\eta_1' = \mu' \xi \sin d$	+ 4.91035	+ 6.19087	+ 6.42881
	$l - l$ A	2.5494	1.2690	1.0312
	B	0.00123	1.29177	1.06984
	$n \cos N = y' - \eta'$	− 7.4609	− 7.4826	− 7.4986

(265)	tan N	0.2403	0.2480	0.2942
	N	+ 119 54 2	+ 119 27 50	+ 116 55 38
	sin N	9.9380	9.9398	9.9502
	log n	+ 7.7632	7.7908	+ 7.8426
(267) N. A.	$l\,l_1$	+ 9.72638	− 8.13767	+ 9.72636
(266) N. A.	$i = \tan f$	+ 7.66686	+ 7.66470	+ 7.66687
(261)	$\zeta = A \cos (B - d)$	9.98024	9.94219	9.83626
	$i\,\zeta$	+ 7.64710	+ 7.60689	+ 7.50313
	$l - l$	B 2.07928	A 0.53078	B 2.22323
	A B	0.00363	0.64291	0.00260
	log L	+ 9.72275	− 8.24980	+ 9.72376
(268)	$M - N$	− 180 30 31	− 250 59 15	+ 1 58 20
	log m	+ 9.72501	+ 7.75435	+ 9.72232
	log $(1 : n)$	+ 2.2368	+ 2.2092	+ 2.1574
	sin $(M - N)$	+ 7.94826	+ 9.97564	+ 8.53675
	cos $(M - N)$	− 9.99998	− 9.51292	+ 9.99974
	m sin ( )	+ 7.67327	+ 7.72999	+ 8.25907
	$\sin \psi = \dfrac{m \sin (M - N)}{L}$	+ 7.95052	+ 9.48019	+ 8.53531
(271)	ψ	{ + 179 29 19	+ 162 24 53 + 17 35 7	+ 1 57 57
(269)	cos ψ	− 9.99998	∓ 9.97922	+ 9.99974
	log (1)	− 1.9595	± 0.4382	+ 1.8809
	(2)	− 1.9618	− 9.4765	+ 1.8795
(271)	Nos. (1)	+ (1) − 91.10	− (1) ∓ 2.74	+ (1) + 76.01
	− (2)	+ 91.58	+ 0.30	− 75.77
	τ	{ + 0.48	− 2.44 + 3.04	+ 0.24
(270)	Greenwich Mean Time T	{ 8^h $2^m.48$	9^h $28^m.56$ 9 34 04	10^h $52^m.24$
(272)	Longitude in time ω	8 0.00 W.		
	Local Astronomical Time T_1	0 2.48	1 28.56 34.04	2 52.24
(273)	Local Civil Time Sept. 9,	12^h $2^m.48$ P.M.	1^h $28^m.56$ P.M. 34 04	2^h $52^m.24$ P.M.
	" " " Middle		1 31 30 P.M.	
	Duration of Totality, double of term (1)		$5^m.48 = 5^h\ 28^s.8$.	

fraction of a minute, or *preferably* to five-place logarithms; but as some of the quantities in the *Almanac* are given only to four places of logarithms, a portion of the work must be computed with this number. With addition and subtraction logarithms small terms in four-place logarithms may be used, and the work continued with five places.

When computing the total phase, the quantities $x' - \xi'$ and $y' - \eta'$ must necessarily be very small, since the observing station is selected near the central line. Four-place logarithms will be suffi-

cient; but the angle N will be a little uncertain, but in the end this does not matter, for τ will be very small. The three columns being computed simultaneously, l and log i for penumbra must be used in the first and third columns, and l_1 and log i_1 for the umbra, in the middle column, which latter will give the times of totality.

The angle ψ has two values as in all the foregoing problems. Here, however, we must take it obtuse, with its cosine negative in the first column for beginning of penumbra, and acute, with its cosine positive for ending in the third column. In the second column the double sign merely counteracts the negative sign of L and take both values, which give the beginning and ending of totality. Quantities for the umbra generally lie so near together that in the first approximation, which may also be the final, one computation answers for both.

CHAUVENET omits to state that the negative sign of L for the umbra of total eclipse necessitates a change of sign of equation (268), and of the first term of equation (269), which omission is here corrected by the condition (271). And in the example this is noted as —(1), referring to the first term of the formula for the total shadow. The two values of ψ after changing the sign give rise to the double sign, the upper for beginning and the lower for ending, as stated above. By this condition τ for beginning of totality properly becomes negative, and for ending, positive. In the example ζ is not computed immediately after η, because it is not needed until equation (267) is reached, when it is taken up. This is one of the few cases in which it is preferable to take up any portion of the formulæ out of the order given in this work. The example given in full will doubtless explain the formulæ more fully.

152. *Second Approximation.*—This may not always be necessary. If the times have not been well chosen to start with, τ may result as a quantity of several minutes, in which case a repetition may be unavoidable, taking as T_0 the resulting times or some integral minute near them; and the results then will be correct within the fraction of a second. In this example it seems that no repetition is necessary. The times were very carefully chosen, and the methods employed will be noticed in Art. 157 on checks to this computation.

The following formula is given by CHAUVENET as a substitute for Nos. (268), (269):

$$\tau = \frac{m}{n} \cdot \frac{\sin(M - N - \psi)}{\sin \psi} \tag{275}$$

which, he remarks, in the second approximation will be more convenient, but when ψ is small will not be so precise.

153. *Angles of Position.*—These are required for the penumbra to guide the observer to that part of the sun's disk where the first contact will appear, so that he may concentrate his attention upon this point to note the earliest moment of the first contact.

Q = Angle of position from the North point toward the East (Art. 80).

V = Angle of position from the Vertex toward the East.

$$Q = N + \psi \tag{276}$$

$$\left.\begin{aligned} p \sin \gamma &= \xi + \tau\xi' \\ p \cos \gamma &= \eta + \tau\eta' \end{aligned}\right\} \tag{277}$$

$$V = Q - \gamma \tag{278}$$

$$= N + \psi - \gamma \tag{279}$$

Or else independently of the eclipse formulæ,

$$\left.\begin{aligned} \sin \zeta \sin q &= \rho \cos \varphi' \sin \vartheta \\ \sin \zeta \cos q &= \rho \sin \varphi' \cos d - \rho \cos \varphi' \sin d \cos \vartheta \end{aligned}\right\} \tag{280}$$

in which ζ = Zenith distance.

q = Parallactic angle = γ of equation (277).

These last formulæ are essentially the same as equation (277).

From the foregoing computation for the final times we have the angles of position from the north point.

(276) $Q = N + \psi$. For beginning, $+ 299° 23' = 60° 37'$ to W.
For ending, $+ 118° 54'$.

EXAMPLE, ANGLES FROM THE VERTEX.

	Beginning.	Ending.		Beginning.	Ending.
(277) τ	+ 9.6812	+ 9.3802	(277) η	− 9.4679	− 9.4287
			η'	+ 4.9103	+ 6.4288
ξ	+ 8.3080	+ 9.8301	$\eta'\tau$	+ 4.5915	+ 5.8090
ξ'	+ 7.6305	+ 7.4897	$l - l\,A$	4.8764	3.6197
$\xi'\tau$	+ 7.3117	6.8699	B	0.0000	0.0001
$l - l\,B$	0.9963	2.9602	$p \cos \gamma$	− 9.4679	− 9.4288
A	0.9501	0.0005			
$p \sin \gamma$	+ 8.2618	+ 9.8296	$\tan \gamma$	8.7939	0.4008
			γ	+ 176° 26′	+ 111° 40′
(278) Angle from the vertex $V = Q - \gamma$				+ 122 57	+ 7 14

The angle from the north point is used in telescopes equatorially mounted, but for others the angle from the vertex is more convenient, which is the point of the sun nearest to the zenith of the observer.

154. *Partial Eclipse; Greatest Eclipse.*—This article is for an eclipse partial at the station of the observer, though it may or may not be total or annular elsewhere. The times are computed for the penumbra exactly as in the foregoing example, but in order to get the middle of the eclipse and the magnitude correctly, it will be necessary to make the first computation for a time as near the middle as possible; or if the times can be known as closely as in the above example, the three columns can be carried on together. A second approximation, however, is a great guard against mistakes. The time of the middle of the eclipse and of greatest eclipse are synonymous. The formulæ are derived in the same manner as those in Art. 78.

$$\tau_1 = -\frac{m \cos (M - N)}{n} \tag{281}$$

$$T_1 = T_0 + \tau \tag{282}$$

The quantities for these equations must be taken from the first approximation, or for a time near the middle; within ten minutes may be near enough. The term for τ, as noted in Art. 78, is already given in the computation. In the above example we have

$$\tau_1 = +\,0^m.30 \qquad \text{G. M. T.} \quad T_1 = 9^h\ 31^m.30$$

The above equations, if applied to the ends of an eclipse, may not give the time of the middle within *fifteen* minutes, nor will the mean of these be anywhere near the correct time.

The middle time as computed for prediction is not equidistant from the times of beginning and ending. This is caused by the rotation of the earth, for when the end of the eclipse is in the afternoon, as in the present example, the surface of the earth at the observing station will be much more inclined to the fundamental plane than at the beginning; the shadow will move more rapidly, and the end occur sooner than otherwise. If an eclipse is computed for the local morning time, the beginning will be nearer to the middle time than the ending.

155. *Duration.*—For total or annular eclipse this is given by doubling the first term of equation (269) for τ taken as positive

$$t = \pm\, 2\,\frac{L \cos \psi}{n} \tag{283}$$

It is also simply the difference between the beginning and ending of totality.

156. *Magnitude.*—The formulæ are

$$\varDelta = \pm m \sin (M - N) = \pm L \sin \psi \tag{284}$$

$$M = \frac{L - \varDelta}{L + L_1} = \frac{L - \varDelta}{2(L - k)} \tag{285}$$

$$k = 0.2723 \tag{286}$$

Or, approximately, omitting the augmentation of the moon's semidiameter,

$$\epsilon = \frac{l}{2(l - k)} \tag{287}$$

$$M = \epsilon \mp \epsilon \sin \psi \tag{288}$$

$\varDelta$ is always to be taken as positive, and M results as a fraction of the sun's diameter. In the last equation the numerical difference is to be used for the double sign. In all these equations the quantities should be taken for some time near the middle to give correct results. After computing $\varDelta$ by logarithms, or taking it from the computation, the rest of the work is more easily performed with natural numbers, the division requiring only three decimals. L has to be computed or taken from the computation. And for a partial eclipse L_1 for the umbral cone has not been used, so that the second form of equation (285) must be substituted. k is the constant used in computing the eclipse tables (Art. 20), and l for equation (287) is given in the BESSELIAN Tables. In formula (288) the numerical difference gives M. These formulæ are simple, and no example is needed for the present eclipse.

157. *Checks on the Times.*—It is stated at the end of the *Nautical Almanac* in the examples of eclipses, that the positive cosine of ψ for beginning and the negative sign for ending will furnish "inaccurate" times for beginning and ending. It is true, but quite as true also that they are utterly worthless for any purposes of check or comparison, for they may differ *fifteen minutes* from the correct times.

A rigorous check upon the times may be had from the simple statement repeated several times in this work,

$$\vartheta = \text{the Local Apparent Hour Angle.} \tag{289}$$

We have the hour angle for the assumed times, ϑ. Correct this for the true times and apply the equation of time, and we have the equality,

$$T = \vartheta_0 + \text{Equation of Time} + \tau. \tag{290}$$

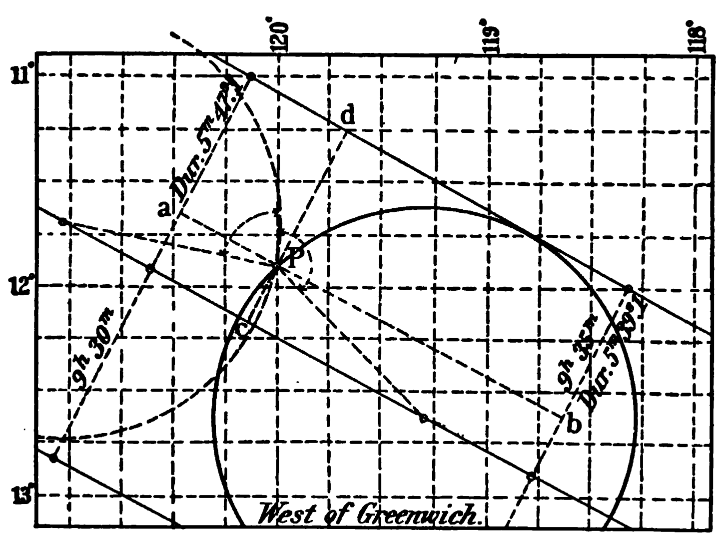
Here the equation of time is to be taken from the *Nautical Almanac*, page I., for the month, and interpolated for the sun's *apparent* hour angle, for which the Greenwich mean time will suffice within a fraction of a second. For the correction of ϑ for the computed times this depends upon the variations of μ_1, which varies 15° per hour of time, or nearly that within small limits, so that the variation of μ_1 will be equal to the change in the time, which is τ. The equation of time is given in seconds, so applying this first and reducing to minutes before applying τ, we have for the present example the following:

CHECK ON THE TIMES BY ϑ, FORMULA (290).

	Beginning.	Middle.	Ending.
Hour angles for the assumed times	11° 11′ 24	23° 26′ 51	43° 42′ 15″
" in time	0^h 4^m $45^s.6$	1^h 33^m $41^s.4$	2^h 54^m $49^s.0$
Equation of Time (*Nautical Almanac*)	—2 45.91	—2 41.17	—2 48.37
Sum	0 1 59.7	1 31 0.2	2 52 0.6
	0 1.99	1 31.00	2 52.01
τ_1 computed	+0 0.49	+0.30	+0.24
Corrected hour angle = ϑ	0 2.48	1 31.30	2 52.25
Computed local times	12 2.48	1 31.30	2 52.24

Fig. 20 (reduced to half size) is a plot of part of the path of this eclipse made on computing paper, to take advantage of the ruled

FIG. 20.



lines; the degrees of latitude and longitude form nearly a square, and though not in proper proportions, yet they serve the purpose

intended. The place from which the above prediction was computed was assumed on this plot, and is marked P. The lines across the path are the positions at $9^h\ 30^m$ and $9^h\ 35^m$, taken from the *Nautical Almanac* and carefully plotted. After the computation was finished, the times of totality were checked as follows: For the middle time with a scale of equal parts (50 to an inch was used) $aP = 58$, $ab = 227$, whence the ratio is 0.255, being a fraction of five minutes, and the time of the middle of the eclipse at P is $31^m.27$. Omitting the decimal, we have the assumed time of the middle for the computation. The correct time is $31^m.30$, which gives the correct fraction of the distance $aP = 0.260$, and interpolating the duration in the *Nautical Almanac* for this fraction we have for the central line $5^m\ 45^s.1$. Now if a semicircle be described in the centre line tangent to the limiting curve, the duration, which is 0 at the limits, will, at the point P, be in proportion to that in the centre line as the cord of the circle is to the diameter. This is easily gotten as follows: Pc measured by scale (50 parts to the inch) gives 31, cd likewise 111; hence the ratio 0.2793, which is, in fact, $\sin \psi$ of the former computation, log sin = 9.4461, from which

		ψ	16° 13′
	$\cos \psi$		9.9824
	log	$5^m\ 45^s.1 = 345^s.1$	2.5379
Duration at P		$5\ \ 31.3 = 331.3$	2.5203
" as computed		$5\ \ 28.8$	
Difference as a check		$0\ \ \ 2.5$	

Having checked the middle time and the duration, the other times of totality are completely checked also. The results are remarkably close for measurements by scale. The closeness of the results of the umbra result greatly from Fig. 20 being drawn on a very large scale, on which a small fraction of a minute can be measured, and partly from great care in plotting, and partly perhaps from chance. For the penumbra great care was taken to locate the point on the chart in the *Nautical Almanac*, and for beginning, taking a scale of six equal parts whose length is a mean of the distance between the 7^h and 8^h and between the 8^h and 9^h curves. For the ending, as the shadow is moving faster, the scale was taken greater than the distance between the 10^h and 11^h curves, which could only be estimated. These means of verification are open to any one who chooses to avail himself of them. No previous computation was made before the example given above.

The lines drawn from P to the centres of the circles make with

the meridian approximately the angles of position of the points of contact of totality, but not exactly, because these angles are measured in the fundamental plane or a parallel to it; whereas the plane of Fig. 20 is tangent to the earth's surface and much inclined, the lower right hand corner being the lowest point.

These angles are as follows, from the example Art. 151; the *negative* sign in equation (268) being used in accordance with the condition (271), which is here shown to be correct.

	Beginning of Totality.	Ending.
N	+ 119 28	+ 119 28
ψ	+ 162 25	+ 17 35
Q	{ 281 53 to E. 78 7 to W.	137 3 to E.

These angles will be again referred to in Art. 165 of the next section. The angle N, which the path makes with the principal meridian, cannot be shown in this figure except approximately on account of the inclination of the plane of the drawing, and also because the meridians are not parallel to the principal meridian.

These angles are usually not wanted for the umbral phase, but are given here for a better understanding of the eclipse.

SECTION XIX.

PREDICTION BY THE METHOD OF SEMIDIAMETERS.

158. THE present section was not included in the original design of the present work, but is an after-thought suggested by the fact that CHAUVENET's treatise gives no idea how the eclipse appears to the eye. Fig. 21 of this section shows the actual relative motions of the observer and the shadow, which are not shown in the previous orthographic projections. And in the latter part of the section, the computation is based upon the supposition of direct vision. For a student who finds CHAUVENET's treatise too difficult, this section will be found an excellent preparatory study. It requires no knowledge of the preceding portions of this work.

For this section the author is chiefly indebted to LOOMIS' *Practical Astronomy*, an excellent work, though this portion of it is not original with him, being the method of the ancient astronomers.

159. *Projection.*—The data for this are simply the elements of the eclipse given in the *Nautical Almanac*, together with the Equation of Time in the body of the *Almanac*. The data for the present example are therefore found in Art. 21 of this work.

The radius of the earth is taken as the difference of the parallaxes of the sun and moon, which has been styled the *relative parallax*. The sun and moon are seen from the earth at their apparent relative sizes, and their *angular* dimensions, their semidiameters, may be taken as linear measurements in seconds to any scale and laid down upon a drawing. The earth and moon, on the contrary, are not seen from the sun according to their *actual* proportions, because they are at different distances, and we must find their *apparent* proportions as seen from the sun. If we regard the cone of solar parallax, its radius x, at the distance of the moon from the sun, will be in proportion to the earth's actual radius, E, as the distances of the moon and earth are from the sun. That is

$$\frac{x}{E} = \frac{r' - r}{r'}$$

And since $r = \frac{1}{\sin \pi} = \frac{1}{\pi}$ and $r' = \frac{1}{\sin \pi'} = \frac{1}{\pi'}$ nearly

$$\frac{x}{E} = \frac{\pi - \pi'}{\pi} \tag{291}$$

$\pi - \pi'$ is then the radius of this cone of parallax at the distance of the moon from the sun, and the moon's radius and this quantity can also be regarded as linear quantities; hence, $\pi - \pi'$ for the earth, s' for the sun, and s for the moon, are now all reduced to the same linear scale. This quantity $\pi - \pi'$ is the actual radius of the earth's sphere used throughout the whole of this work, which can easily be shown from equation 218, Art. 113, namely—

$$y = \frac{1}{1 - b} \frac{\delta - \delta'}{\pi} \tag{292}$$

This equation gives y at the time of conjunction as a decimal fraction, of which the denominator must be unity, the radius of the earth; and the value of b we have from Art. 25, therefore the denominator becomes

$$\pi(1 - b) = \pi\left(1 - \frac{\sin \pi'}{\sin \pi}\right) = \pi\left(1 - \frac{\pi'}{\pi}\right) = (\pi - \pi') \tag{293}$$

which agrees with the quantity gotten above.

160. *Data for the Projection.*—The quantities following must be taken from the *Nautical Almanac* and reduced as here shown. They are the elements of the eclipse, given also in Art. 21 of the present work. The nearest second of arc is close enough.

G. M. T. of ☌ in R. A.	8^h	49^m	34^s
Longitude of the place (west)	8	0	0
Local Astronomical Time	0	49	34
" Civil Time	12	49	34
Equation of Time (Mean to Apparent)		+ 2	47
Local Apparent Time	12	52	21
Sun's Declination δ' or d	+ 5°	14′	56″
Moon's Declination δ	+ 5	4	31
Moon *south* of the sun in this example	0	10	25
Difference of Hourly Motions in R. A. $137^s.16$ =		34	17

This is to be reduced to the arc of a great circle by multiplying it by the cosine of the moon's declination, which gives 2049″.

Difference of Hourly Motions in Declination — 636″

Moon's parallax, 3683″, to be reduced to the latitude of the given place, — 11° 56′, by multiplying it by the earth's radius for the place. This may be taken from Table IV. In the present example the earth's radius is 9.9999, which hardly changes the value.

Hence, the Reduced Parallax	3682″
Sun's Parallax	9
Relative Parallax	3673.
Sun's Semidiameter	953.
Moon's Semidiameter	1004.

Geographical latitude of given place is — 11° 54′, which must be reduced to the geocentric latitude, as in Art. 149, or the angle of the vertical may be numerically subtracted, taken from Table XII.

Hence, the Reduced Latitude for the Place, 11° 49′

161. With the relative parallax 3673″, and assuming a scale of 1000″ to an inch, describe the circle $ADBC$ (Fig. 21, Plate VIII.), which will represent the earth's sphere. AB and CD are the principal axes, as in the former projections. The earth is supposed to be seen from the sun, which is therefore vertically over the centre of the circle Z at apparent noon; and this therefore determines the positions of the parallels of latitude on the earth. It is also very nearly vertically over any other part of the drawing.

The path of the moon will now be drawn. As it is 625″ south of the sun at conjunction, set off this amount, ZK, below the point Z, on the axis of Y, which is the meridian of conjunction in right ascension at local apparent noon. Lay off ZH along the axis of X equal to 2049″, the difference of the hourly motions of the sun and moon in right ascension reduced to the arc of a great circle, and HI perpendicular to it and equal to $-636''$, the difference of the hourly motions in declination. ZI will then represent the direction of the moon's path and the distance of its motion in one hour. Through K draw an indefinite line parallel to ZI, which will be the actual path of the moon's centre.

The hour points on the path must be so laid off that the time of conjunction at *apparent* noon; that is, $52^m\,21^s = 52^m.35$ after the 12-hour point, shall fall on the point K. ZI measures by scale 2146″, which is the motion in one hour, or 60′, so we make the proportion

$$60^m.00 : 52^m.35 :: 2146 : 1872$$

Laying off the distance, 1872, of the scale toward the left of K gives the 12-hour point of the path. The other hours are found by laying off on each side of this the distance, ZI. The path can now be divided into 10-minute spaces, and into minutes where necessary.

162. The parallel of latitude of the given place is now to be projected, which will be an ellipse. The axes may be found in two ways, the first of which is wholly geometrical. Revolve the north pole around the line CD as an axis down into the fundamental plane; the south pole will fall on the left of the axis at P'. The semicircle CAD will fall into the right line CZD. The right line CZD will become the semicircle CBD, the sun being then in the line ZB, produced; and its declination being *north*, the equator will become a right line, $E'E''$, with the arc BE', equal to the sun's declination, the equator, $E'E''$, will be perpendicular to the axis ZP', and the given parallel will be the line ab, 11° 49′ south of the equator. Now revolve the axis of the earth to its normal position. P' will fall below the fundamental plane out of sight. E' will fall at E, and the ellipse AEB may be drawn if desired; the point b of the given parallel will fall at d on the upper surface of the sphere, while a will fall at c on the under surface; cd is the conjugate axis of the ellipse; bisecting this e is the middle point, which is also the revolved position in which this parallel intersected the axis of the earth. Through e the transverse axis mn can be drawn at right

angles to $c\,d$; the length of the semiaxis is given by the half length of the diameter $a\,b$, which is shown in its true length. Upon these axes the ellipse can be described.

The second method is more accurate, as the axes are partly computed. It is seen from the geometry of the figure that $Z\,d$ is in projection the angle $\varphi' - d$ and $Z\,c$ the angle $\varphi' + d$, and that the half length of $a\,b$ is $\sin\varphi$, each in proportion to the radius $Z\,B$. Hence, we have

$$\left.\begin{array}{l}\text{Distance of the two ends of the conjugate axis from the centre} \\ \qquad\text{of the sphere,} \qquad \rho \sin(\varphi' \pm d)\ ^* \\ \text{Length of semitransverse axis,} \quad \rho \cos\varphi \\ (+ \text{ for the northern point, } - \text{ for the southern, in all cases.})\,.\end{array}\right\} \quad (294)$$

d is the sun's declination, or, more properly, the quantity used in the eclipse formulæ, whose sine and cosine are given in the BESSELIAN tables of the *Nautical Almanac.*

In the present case we have

Northerly point	$Zc = 3673 \sin(-\ 6^\circ\ 44') =$	$-\ 431''$
Southerly point	$Zd = 3673 \sin(-16\ \ 54\ ) =$	-1068
Semitransverse axis	$mn = 3673 \cos(-11\ \ 49\ ) =$	3595

From these figures the ellipse can be constructed. The best method to be used in this case is by drawing two circles, one upon each axis. From any point (f) of the outer circle draw a line to the centre, cutting the smaller circle at the point g. From f and g draw lines parallel to the axes, and where they intersect at h will be a point of the ellipse; any number of points can be gotten in the same manner, and the advantage of this construction is that if the outer circle is divided into six equal portions of 15° each, these points will give the hours 1, 2, 3, etc., before and after the noon point on the axis. Intermediate points can be gotten if desired, for the divisions for equal times will be unequal on the ellipse.

This projection, depending upon the true sun, gives local apparent time; d is, therefore, the position of the sun at *apparent* noon, and the integral hours can be laid off by the method described above. 6^h A. M. will be the point m and 6^h P. M. the point n; sunrise and sunset will be the points where the ellipse is tangent to the earth's circle; in the present case very near m and n. The ellipse can be divided into 10-minute spaces for that portion along which the eclipse occurs.

* *Treatise on the Projection of the Sphere, Including Orthographic and Stereographic Projections*, by the author of the present work, p. 12.

163. The times can now be found as follows: Take the sum of the semidiameters 1957″, in the dividers, and running the left leg on the moon's path, and the right leg on the ellipse, find the points where they both mark the same time; this is the beginning of the eclipse. Then move the left leg on the ellipse and the right on the path until they also mark the same times, which give the ending. With the difference of the semidiameters 51″ find in the same manner the beginning and ending of totality. The times thus measured are found to agree with the computed times as near as a drawing can give them. They are reduced to mean time by applying the equation of time (apparent to mean time).

	Apparent Time.	Equation of Time.	Mean Time.
Eclipse begins	$12^h\ 5^m$.	$-0^m\ 2^s.8$	$12^h\ 2^m$
Totality begins	1 31		1 28
Totality ends	1 37		1 34
Eclipse ends	2 55		2 52

This projection requires the *utmost exactness* in all its parts, especially in the angle of the moon's path, and the projection of that portion of the ellipse covered by the eclipse. The times of totality will be found especially difficult. All the quantities that can be should be computed similarly to those for the ellipse, so as to reduce the error of drawing. The drawing for Fig. 21 was made double the size of the figure and reduced to lessen the errors of drawing.

164. From each of the four points just determined on the ellipse, with the radius of the sun's semidiameter 953″″ describe circles; and from the corresponding points of the moon's path, with the radius of the moon's semidiameter 1004″ also describe circles. The pairs from the corresponding points should be tangent to one another, and show the relative positions of the sun and moon at the times of the four contacts.

165. *The Angles of Position.*—From the centre of the sun for beginning and ending draw lines through the point of contact to the centre of the moon. The angles which these lines make with a line parallel to the axis of *Y* through the centre of the sun's disk will mark the angles of position in the disk to the point of contact, measured from the north point toward the *celestial* east as positive. This is the direction in which the sun moves in the celestial sphere, from west to east.

The angles as computed in Art. 153 are 60° to the west, negative, and 119° to the east, positive; they are marked in Fig. 21, and can be measured with a protractor. The first is negative and the second positive, because the moon moving the faster, overtakes the sun making the first contact on the *western* limb, and the last on the eastern limb.

For the total phase, if we likewise consider the line from the centre of the sun through the point of contact to the centre of the moon, we see that it has doubled back upon itself, showing that the angles of position in projection for the umbral phases are given by the formula.

$$\left.\begin{array}{ll}\text{For Total Eclipse} & Q_1 = N + \psi \pm 180°. \\ \text{For Annular Eclipse} & Q_1 = N + \psi.\end{array}\right\} \qquad (295)$$

From Art. 157 we have from the computation for totality:

Q	Beginning + 281° 53′ to E.	Ending + 137° 3′ to E.
Constant	± 180 0	± 180 8
Q_1 for Projection	+ 101 53 to E.	− 42 57 to W.

These angles are also shown on Fig. 21, the positive being measured toward the *left*, and the negative to the right, on account of the change of 180° made above.

If we consider the limbs of the sun on which the contacts take place, we find them to be as follows:

		Total Eclipse.	Annular Eclipse.
First	Eclipse begins	Partial begins to W.	Partial begins to W.
Second	Umbra begins	Totality begins to E.	Annulus formed to W.
Third	Umbra ends	Totality ends to W.	Annulus ruptured to E.
Last	Eclipse ends	Partial ends to E.	Partial ends to E.

We notice here that the change to the east limb of the sun for direct vision, the change of direction, formula (295), in the drawing, and the negative sign of l in the calculation, all mean the same thing for a total eclipse, and are the contrary to the other phases.

166. *Partial Eclipse.*—If the eclipse should not be total at the given place on the ellipse, find the points of nearest approach which mark the same times on the path and ellipse; the line joining them will be at right angles to the path of the moon, and can be found by moving a small right-angled triangle, keeping one side on the path, and finding where the other edge cuts the same time on the ellipse. From these points, drawing circles as before, the appearance of the sun at greatest eclipse is shown, and the distance can be measured by

scale on the cross line; the magnitude is the portion of the sun's disk covered by the moon divided by the diameter. The time of greatest eclipse above given can be reduced to mean time by applying the equation of time.

167. This method of finding the times for any place may also be applied to Figs. 1 and 2 in exactly the same manner as in this section; the passage of the umbral or penumbral shadow over the place will determine the times; l and l_1 are given in decimals of the sphere's radius as unity. For greater accuracy, l or l_1 can be reduced to L and L_1 by formulæ (266) and (267) for prediction. This takes account of the augmentation of the moon's semidiameter.

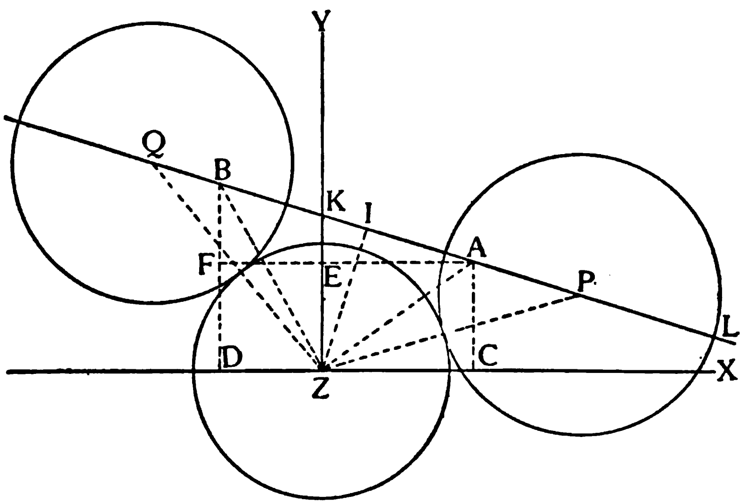
168. *The Computation.*—This method takes no account of change in the data by variable quantities, so that the error of a first approximation is probably greater than in the other method; and for a second approximation the times must be assumed to the nearest minute if possible. Several preliminary reductions are necessary, and when the computation is correctly made, the results are correct within a fraction of a second, which is all that can be required.

The time of conjunction in right ascension must be found at least approximately, and two consecutive hours selected which include this time. For these two hours the right ascensions, declinations, semidiameters, and parallaxes of the sun and moon must be interpolated from the *Nautical Almanac* to the tenth of a second of arc. The moon's right ascension and declination are to be corrected for parallax, and the declination also corrected for the convergence of the meridians to be explained presently. The moon's apparent semidiameter is to be corrected for the augmentation; this, however, may be omitted in the first approximations, but it has great influence upon the times of the total phase or annular eclipse.

Maps of the earth's surface are usually drawn with the north at the top, and the shadow of the moon in a solar eclipse will be shown as moving from the left toward the right. If now we face the sun toward the south, the moon will be seen to encroach upon the sun on the *right* hand, and to move from *right* to *left;* and this method by semidiameters, being founded upon their visual contact, their relative positions must be represented as they would be seen to the eye. Therefore, in Fig. 22, let Z represent the centre of the sun, supposed to be at rest; A the position of the moon at the first hour, and B at the second hour, moving from right to left. ZC will then repre-

sent the difference of the right ascensions for the first hour and ZD for the second hour. These will be noted by x as usual, and CA and BD, the differences of declinations, by y. The hourly motions are $AF = x'$ and $FB = y'$. And here it may be noted that AF is supposed to be a perpendicular to the meridian YZ at E, and if a parallel of latitude be drawn through A, it will cross the meridian YZ *below* the point E; the declination of A is therefore less than that of E, so that a correction must be applied to the moon's declination to give the point E correctly. The declination at B for the second

FIG. 22.



hour must likewise be corrected, and these depend upon the difference of right ascension, ZC and ZD. The first hour assumed is considered as T_0, the other merely gives the differences of right ascension and declination and the hourly motions.

Angles measured toward the left are here considered as positive, and distances toward the left are also positive. Let

The differences of right ascension,	$\alpha - \alpha' = x$
" of declination,	$\delta - \delta' = y$

Then let $YZB = M$ positive and $ZB = m$. Then

$$m \sin M = x \qquad\qquad m \cos M = y$$

Also let the angle which the line of the path makes with the meridian $YKB = N$, and $AB = n$. Then

$$n \sin N = AF = x' \qquad\qquad n \cos N = FB = y'$$

x' and y', being the hourly motions of x and y, they are also in figures the differences of the hourly motions of the sun and moon at the assumed hours. N, as in the former method, is always positive, and never deviates more than about 30° greater or less than 90°. In the figure it is less than 90°, and n is the motion of the shadow in one hour $= AB$. The angle KBZ, or the supplement of $KAZ = ZAL$, is $M - N$, and the line ZI, perpendicular to the path, is $m \sin(M - N)$, and the angles at P and Q, taken obtuse for beginning and acute for ending, are

$$\sin \psi = \frac{m \sin (M - N)}{s' + s}$$

in which $s' + s$, the sum of the semidiameters, is either of the lines ZP or ZQ.

We also have $\qquad PI = IQ = (s' + s) \cos \psi$

and $\qquad AI = m \cos (M - N)$

Hence $\qquad PA = PI - AI$ and $AQ = IQ + AI$

or
$$\begin{cases} PA = (s' + s) \cos \psi - m \cos (M - N) \\ AQ = (s' + s) \cos \psi + m \cos (M - N) \end{cases}$$

and the time of describing these lines is found by dividing them by the motion in one hour, which is n.

PA is properly negative, which will result if ψ is taken as an obtuse angle for beginning. These quantities, noted as τ when divided by n, are to be subtracted from and added to the epoch hour A to give the times.

For a second approximation two times being assumed near the resulting times, will each become T_0, and may be represented in the figure by A and B, so that the final distances will be

$$AP = PI - AI$$
$$BQ = IQ - IB$$

For the total phase the path approaches so near the centre of the sun Z that with the *difference* of the semidiameters a figure very similar to Fig. 22 is produced on a smaller scale, and the same reasoning and formulæ are applicable, substituting $s' - s$ instead of $s' + s$. This will give the shadow for total eclipse negative, as in CHAUVENET's discussion. ZP and ZQ will then represent the difference of the semidiameters $s' - s$ instead of their sum.

In Fig. 22 we can show the triple angle of CHAUVENET's formula (275) of the previous section. For beginning, in the two right-angled triangles IZP and IZA, it is the difference of the angles at Z. The

angle ψ at P is obtuse, and the acute angle is $180 - \psi$, and in the triangle IZP the angle at Z is then $-90 + \psi$.

In the triangle IZA $M - N$ is the obtuse angle at A, the acute angle being $180 - (M - N)$, and the angle at Z is $-90 + (M - N)$.

Hence AZP is the difference of these angles.

Hence $AZP = IZA - IZP = M - N - \psi$ toward the left as positive.

For ending in the two right-angled triangles PZI and QZI, it is the difference of the angles at Z.

In the first the angle at Z is $90 - (M - N)$, and in the second $90 - \psi$.

Hence $BZQ = IZQ - IZB = -\psi + M - N$ toward the left as positive.

169. Formulæ deduced above are as follows:

Notation:

- α' δ' Sun's right ascension and declination corrected for parallax.
- α δ Moon's R. A. and Dec. corrected for Parallax and the declination corrected for convergence of meridian.
- π Moon's parallax reduced for the latitude of the place. This value is to be used for the corrections in R. A. and Dec.
- s Moon's true semidiameter, constant of irradiation deducted, and corrected for augmentation.
- s' Sun's true semidiameter, constant of irradiation deducted.
- φ' The geocentric latitude of the given place.
- x' The difference of the hourly motions of the sun and moon reduced to the arc of a great circle.
- y' The difference of the hourly motions in declination.
- ϑ Moon's true hour angle.

Preliminary reductions for the above quantities.

Moon's true hour angle.

$$\left.\begin{aligned}\vartheta &= \text{sidereal time} - \text{Moon's true R. A.}\\ &= \theta - \alpha.\end{aligned}\right\} \tag{296}$$

Geocentric latitude of the given place

$$\left.\begin{aligned}\tan\varphi' &= \sqrt{1 - e^2}\tan\varphi\\ \text{or}\quad \varphi' &= \varphi - \text{the angle of the vertical.}\end{aligned}\right\} \tag{297}$$

Relative Parallax $= \pi - \pi'$ from *Nautical Almanac.* (298)

Reduced parallax for the latitude of the place.

$$\Pi = (\pi - \pi') \times \rho \text{ for the given place.} \tag{299}$$

Parallax in right ascension.

Compute $$a = \frac{\sin \Pi \cos \varphi'}{\cos \delta} \tag{300}$$

And when the apparent hour angle affected by parallax is known

A $$\sin \pi_1 = a \sin (\vartheta + \Pi) \tag{301}$$

When the *true* hour angle ϑ is known.

B $$\tan \pi_1 = \frac{a \sin \vartheta}{1 - a \cos \vartheta} \tag{302}$$

C $$\pi_1 = \frac{a \sin \vartheta}{\sin 1''} + \frac{a^2 \sin^2 2\vartheta}{\sin 2''} + \frac{a^3 \sin 3\vartheta}{\sin 3''} \tag{303}$$

This correction always *increases* the hour angle numerically.
Parallax in declination.

Compute $$\cot b = \frac{\cos (\vartheta + \frac{1}{2}\Pi) \cot \varphi'}{\cos \frac{1}{2}\Pi} \tag{304}$$

$$c = \frac{\sin \Pi \sin \varphi'}{\sin b} \tag{305}$$

When the apparent declination affected by parallax is known

D $$\sin \pi_2 = c \sin (b - \delta + \Pi) \tag{306}$$

When the true declination is known

E $$\tan \pi_2 = \frac{c \sin (b - \delta)}{1 - c \cos (b - \delta)} \tag{307}$$

F $$\pi_2 = \frac{c \sin (b - \delta)}{\sin 1''} + \frac{c^2 \sin 2(b - \delta)}{\sin 2''} + \frac{c^3 \sin 3(b - \delta)}{\sin 3''} \tag{308}$$

This correction *increases* the distance of the moon from the north pole.

Correction for moon's declination for convergence of the meridians:

$$\left.\begin{aligned} \Delta\delta &= (a - a')^2\, 900 \sin 1'' \sin 2\delta \\ &= [7.6398]\ (a - a')^2 \sin 2\delta \end{aligned}\right\} \tag{309}$$

$a - a'$ is the difference of the right ascensions of the sun and moon, and $\Delta\delta$ the correction in seconds of arc, to be numerically added, whether δ is north or south.

Augmentation of the moon's semidiameter.

When the moon's apparent altitude is known, and the parallax in declination π_2 is known,

$$\Delta = s \sin \pi \cot (b - \delta) - \tfrac{1}{2}\, s \sin^2 \pi \tag{310}$$

b is the quantity in equation (304) above given, and $(b - \delta)$ is used in equations (307–8). This quantity is to be added to the moon's semidiameter.

The hourly motions in right ascension reduced to the arc of a great circle

$$x' = (\Delta a - \Delta\delta) \cos\delta \tag{311}$$

In LOOMIS' *Practical Astronomy* are given tables by which several of the above quantities may be taken out at once. The correction for the sun's parallax in formulæ (298) is small, so that it is here included with that of the moon, and both reduced together. In the reduction for φ', the radius of the earth for the latitude is to be used. This may be taken from Table IV. or the angle of the vertical interpolated. For the parallaxes in right ascension and declination several formulæ are given, either of which may be used, according to circumstances. The semidiameters of the sun and moon given in the body of the *Almanac* are the *apparent* values, which contain constant terms for irradiation, as explained in Arts. 20 and 21. These constants must be deducted first, then the moon's semidiameter corrected for augmentation.

Principal formulæ, Fig. 22,

$$\left.\begin{array}{ll}\text{Differences of R. A.} & x = (a - a') \\ \text{Differences of Dec.} & y = (\delta - \delta')\end{array}\right\} \tag{312}$$

$$\left.\begin{array}{l} m \sin M = x \\ m \cos M = y \end{array}\right\} \tag{313}$$

$$\text{Hourly motions,} \quad \left.\begin{array}{l} n \sin N = (\Delta a - \Delta a') \cos\delta \\ n \cos N = (\Delta\delta - \Delta\delta') \end{array}\right\} \tag{314}$$

$$\sin\psi = \pm \frac{m \sin (M - N)}{s' \pm s} \tag{315}$$

$$\tau = \pm \frac{(s' \pm s) \cos\psi}{n} - \frac{m \cos (M - N)}{n} \tag{316}$$

$$T = T_0 + \tau \tag{317}$$

Angles and distances measured from right to left are positive. ψ is to be taken as obtuse for beginning with its cosine negative, and acute for ending with its cosine positive. In equations (315) and (316) the upper sign of $(s' \pm s)$ gives the times of a partial eclipse, and the lower the total or annular eclipse. For a total eclipse *only*, the lower signs before these equations are to be used to counteract the negative of this factor in equation (315).

Equation (316) may be put into another form giving the triple angle. Multiply numerator and denominator by $\sin\psi$.

$$\tau = \pm\frac{\sin\psi\,(s' \pm s)\cos\psi - m\cos(M-N)\sin\psi}{n\sin\psi}$$

Substitute the value of $\sin\psi\,(s' \pm s)$ from (314),

$$\tau = \frac{m\sin(M-N)\cos\psi - m\cos(M-N)\sin\psi}{n\sin\psi}$$

$$\tau = \frac{m}{n}\cdot\frac{\sin(M-N-\psi)}{\sin\psi} \tag{318}$$

In LOOMIS' *Astronomy* this equation is given thus,

$$\tau = \frac{m}{n}\cdot\frac{\sin[\psi \mp (M-N)]}{\cos\psi} \tag{319}$$

which results from angles being considered numerically, without regard to sign, and also because N is taken as acute; $= 90 \pm N$ of CHAUVENET'S notation.

In using these formulæ, at least one decimal of arc should be retained after making the reductions, which will give five figures for the quantities in seconds considered as linear; and compute with five-place logarithms to seconds of arc. Signs must be regarded in the above formulæ, and the angles of position are found as in the former section. In a partial eclipse the middle time is given by the second term of equation (316), and the magnitude can be computed as in Art. 156.

SECTION XX.

SHAPE OF THE SHADOW UPON THE EARTH.

170. *Shadow Bands.*—My attention was first called to the subject of this section during the total eclipse of the sun, May 28, 1900, while at Newberry, South Carolina, where I went with Professor Cleveland Abbe and Professor Frank H. Bigelow, of the Weather Bureau, to observe that eclipse.

Just before and just after the totality of a solar eclipse, certain lines of light and shade are sometimes seen on the ground, moving perhaps in a direction at right angles to their length. They have

been recorded of various widths from one to three inches, or sometimes more, with spaces of light generally greater, even two or three times their width; their motion variable, generally from two to six feet per second, occasionally much greater. These appearances have been denominated *Shadow Bands*, and their origin is unknown, but has been supposed to be due to the diffraction of the sun's light. More probably they are caused by the undulations and disturbances of the density of the atmosphere within the cone of shadow caused by the lower temperature within the cone, which may fall some five or six degrees, thereby producing intermittent opacity. This theory has been proposed by Professor BIGELOW, based upon the voluminous observations made by the Weather Bureau during this eclipse.

These observations were made over a tract of country extending about six hundred miles on each side of the centre line, and over the whole of this region a decided fall of temperature was recorded, commencing at each station on the centre line about three-quarters of an hour before totality, the minimum being reached just at totality, but the normal temperature not being reached until about two hours after totality. At the distant stations the fall was not quite so great nor the interval of low temperature quite so long. If the shadow bands are caused by the fall of temperature as above noted, it seems to me that they must exist throughout the whole of this region. Within the umbral cone there is not sufficient light to cast a shadow, but they are seen in the dim light just before and after totality, and the brighter light overpowers the shadow, so that the bands can be seen only just before and just after totality, when the light is dim.

The directions for observing the shadow bands are, in brief, to fasten a sheet upon the ground stretched free from wrinkles, the sides being north and south. When the bands are first seen, lay a lath or straight stick along their length and let it remain; estimate how many bands and spaces there are to one foot. At the close of totality proceed in the same manner, laying another stick parallel to their length, which will generally be in quite a different direction from the first. After the shadow bands cease to be seen, measure the directions of the sticks carefully with a compass.

Whatever may be the cause of the shadow bands, since they are seen only around the shadow of total eclipse, it is important in any discussion of them to know the contour of the shadow upon the ground, or, in other words, the direction of the tangent line at any place where they may have been observed. It is not known to the author that this problem has heretofore been worked out.

171. *The Ellipse of Shadow.*—The shadow of a total eclipse as it appears upon the surface of the earth, disregarding local irregularities, is the intersection of a sphere and cone, the compression of the earth being here neglected. An oblique section of a cone by a plane is an ellipse, but no plane can cut an ellipse from a sphere. The intersection is therefore a curve of double curvature, but since the cone is quite small compared with the sphere of the earth, the small portion of the earth approaches to a plane, so that the curve approaches to an ellipse, and practically *is* an ellipse, as will be seen below.

The minor axis b of this ellipse is evidently the radius of the shadow at the height of the given place above the fundamental plane, which, for a point of the central line of shadow, is the quantity L in the computation for *Duration*, Section XIV., Arts. 116 and 117 ; but being given there in parts of radius, it must be divided by the sine of one minute to reduce it to minutes of arc. The axis of the cone of shadow during totality is very nearly the line from the observer (anywhere within the central path) to the sun, and the radius, L, is measured at right angles to this axis. The surface of the earth at this point, being the horizon of the observer, is inclined to the axis of the shadow by the zenith distance ζ of the sun, and the oblique projection of L on the plane of the earth's surface is b divided by $\cos\zeta$, which is a, the major axis of the ellipse of shadow. The zenith distance is usually computed for the given place from the geographical zenith, but if computed for the place of the centre of the ellipse for the geocentric zenith, it is seen to be the quantity $\cos\beta$ already made use of in the computation for the Central Line, Section XII., Arts. 108 and 111, so that, substituting $\cos\beta$ for $\cos\zeta$, we have the quantity λ used in the Limiting Curves of Total Eclipse, Section XVI., Arts. 136 and 138.

One more element is required to determine completely the ellipse, and that is the direction of the major axis, which is evidently in the plane between the sun and the zenith of the place. Assuming this place to be the centre of the ellipse, this line on the earth's surface is the azimuth of the sun, which is usually measured from the south point.

Collecting these formulæ from the sections above named, together with those for azimuth A and zenith distance ζ, we have in full :

$$\vartheta = \mu_1 - \omega \tag{320}$$

$$\left.\begin{aligned} \sin\zeta \sin A &= \cos d \sin\vartheta \\ \sin\zeta \cos A &= -\cos\varphi \sin d + \sin\varphi \cos d \cos\vartheta \\ \cos\zeta &= \sin\varphi \sin d + \cos\varphi \cos d \cos\vartheta \end{aligned}\right\} \tag{321}$$

$$\cos\beta = \cos\zeta \text{ nearly,}$$

or else compute $\sin\beta \sin\gamma = x$

$$\sin\beta \cos\gamma = y_1 = \frac{y}{\rho_1} \qquad \log \rho_1 = 9.9986 \tag{322}$$

$$L = l_1 - i \cos\beta \tag{323}$$

$$\lambda = \frac{L}{\cos\beta \sin 1'} \tag{324}$$

Angle of the transverse axis from the north point.

$$= 180^\circ - A \tag{325}$$

$$\text{Transverse axis } a = \lambda \tag{326}$$

$$\text{Conjugate axis } b = \frac{L}{\sin 1'} \tag{327}$$

By these formulæ the axes can be computed *de novo*, d x y l_1 i μ_1 being given in the eclipse tables of the *Nautical Almanac.* The sun's declination δ' should be used in place of d if the azimuth is computed for any place *not* on the centre line of the path of shadow. They differ but little, and one may be used for the other without much error, as none of the quantities is required closer than one or perhaps two decimals of a minute of arc, using only four-place logarithms. a and b will result in minutes of arc, which is taken as the unit of measure in this section. Angles are measured to the nearest minute of arc, and these will be sufficiently exact. The direction of shadow bands cannot be measured closer than one degree of arc.

If this section should be of use to astronomers, the quantities L and λ, the axes of the ellipse of shadow together with $\cos\beta$, the zenith distance can be furnished by the *Nautical Almanac* office, for they are already used in the calculations, or they can be given for every ten minutes in the special pamphlet usually published in advance of any total eclipse visible in the United States.

172. *The Axes and Conjugate Diameters.*—It remains to show that the above-mentioned lines a and b are really the axes of the ellipse of shadow. If the ellipse is centred upon one of the five-minute points, for which the centre line and limits are given in the *Nautical Almanac*, the line joining these two points may be regarded as a conjugate diameter of the ellipse, since on the earth's surface at its ex-

tremity the limiting line is tangent to the shadow. On the fundamental plane this line is determined by the quantity Q, so taken as to be at right angles to the direction of the motion of the shadow; the tangent at its extremity is therefore also parallel to the direction of the motion. It therefore occurred to me that if another computation were made using the formulæ for limits, but taking Q differing 90° from the former value, we should obtain the other conjugate diameter of the circle on the fundamental plane, the tangent at the extremity of each being parallel to the other line. And in projection upon the plane of the earth's surface, since parallel lines will also be parallel in projection, these lines and tangents will be conjugate diameters of the ellipse of shadow. This reasoning is found to be correct, and the proof of the ellipse as well as of the axes and conjugate diameters is given by the property of the ellipse, that the sum of the squares of the axes is equal to the sum of the squares upon any set of conjugate diameters.

The shadow of a carriage wheel on the ground is a good though homely illustration of this theorem. A cane placed on the tire in the plane of the wheel and perpendicular to one of the spokes will show in shadow a conjugate diameter if there is another spoke parallel to the cane. If the conjugate diameters can be so selected that they and their shadows are at right angles, they will be the axes of the ellipse of shadow.

In this example we might have shown this proof of the point of the centre line and limits given in the examples of Sections XII., XIV., and XVI.; but as the quantities L and λ will be needed for other problems in connection with the Given Place of Section XVIII., we here compute another point of the central line, duration and limits, at 9^h 30 . The work for the two first of these is given in brief, but sufficient to show the correctness of L and λ, which are the axes of the ellipse a and b. The work for limits is given in full and headed "Limits for b'." It is identically the computation for the limiting curve given in the *Nautical Almanac*. The resulting geographical positions are the ends of the line we have chosen as a conjugate diameter, b'. In addition to this, the last column gives a similar computation to the previous and computed by the same formulæ, except that Q is *increased* by 90°, making an obtuse angle. The resulting geographical positions of this latter are on the centre line, and are also the two extremities of the other conjugate diameter, a'. In the present case a' is here greater than b', but this may not always be so if the axis b is greater than a.

The computation for the axes and conjugate diameters is given in the following example:

Axes and Conjugate Diameters ($9^h\ 30^m$).

Axes a and b.

Centre Line in brief.

		9^h 30
N. A. x	+0.39586	+9.57503
N. A. y	—0.28685	—9.45766
(206) (207) y_1		9.45911
tan γ		0.11592
γ		+127 26 33
sin γ		9.89980
sin β		9.67523
(208) cos β		9.94490
(209) ϑ		+ 22° 35′ 16″
N. A. μ_1		143 11 51
ω		120 36 35
tan ϕ_1		—9.32307
(210) tan ϕ		—9.32454
ϕ		—11° 55′ 18″

Duration.

(219) N.A. l_1	—0.013725	—8.1375
N. A. log i		+7.6647
log i cos β		+7.6096
$l - l\,A$		0.5279
B		0.6407
log L		—8.2503

The axes a and b.

(219) L : cos β		—8.3054
sin 1		6.4637
(326) $a = \lambda$	69.45—	1.8417
(327) $b = L$: sin 1′	61.18	—1.7866

Conjugate Diameters a' and b'.

		Limits, for b'. 9^h 30		New point, for a'. 9^h 30
(250) λ		—1.8417		—1.8417
From Duration sin Q		+9.6930	sin$(Q+90°)$	+9.9395
h sin H = cos Q		+9.9395		—9.6930
N. A. sin d_1		+8.9620		+8.9620
(251) h cos H = sin Q sin d_1		+8.6550		+8.9015
tan H		1.2845		0.7915
H		87 1 36		—80 49 9
sin H		+9.9994		+9.9944
(251) log h		+9.9401		+9.6986
ϑ from Central $\vartheta - H$		+64 26 20		+103 24 25
sin $(\vartheta - H)$		9.9553		+9.9880
cos $(\vartheta - H)$		9.6350		9.3652
(252) $d\phi$	+54.589	+1.7371	—33.75	—1.5283
(252) log (1)	+5.494	+0.7399	— 1.693	+0.2286
N. A. cos d_1		+9.9982		+0.9982
log (2)	—34.112	1.5329	—60.172	—1.7794
$d\omega$ = (1) + (2)	—28.618		—61.865	

The geographical positions.

From the first column, Limits.

	ϕ	ω
(253) North Limit	—11° 0′.7	120° 8′.0
South Limit	—12 49 .9	121 5 .2

From second column. Points on Centre Line.

(328) Point preceding (Easterly)	—12° 29′.1	119° 34′.7
Point following (Westerly)	—11 21 5	121 38 .5

As Chauvenet gives a criterion for the additions $\varphi + d\varphi$ and $\omega + d\omega$, we have likewise for the geographical positions resulting from the last column:

$$\left.\begin{array}{l}\text{The sign of the first term for } d\omega \text{ must be changed, and}\\ \text{The Point preceding is given by } \varphi + d\varphi \text{ and } \omega + d\omega\\ \text{The Point following is given by } \varphi - d\varphi \text{ and } \omega - d\omega\\ \text{For an Annular Eclipse reverse these conditions.}\end{array}\right\} \quad (328)$$

No further use is here made of the latitudes and longitudes, as it is only the quantities $d\varphi$ and $d\omega$ which we now require.

173. *Proof of the Ellipse.*—We first require the length of the conjugate diameters. If we regard the spherical triangle between the point on the centre line, the point of the limits, and the pole, we have the angle B at the centre line and the line b required.

The usual formulæ are rigorously

$$\begin{aligned} \sin b \sin B &= \cos \varphi \sin (\omega - \omega') \\ \sin b \cos B &= \sin \varphi' \cos \varphi - \cos \varphi' \sin \varphi \cos (\omega - \omega') \end{aligned} \qquad (329)$$

But as $(\omega - \omega')$ is a small angle, we may write the arc for the sine and place the cosine equal to unity; writing the arc for the sine, also for b and $(\varphi - \varphi)$ in the second equation, we have simply

$$\left.\begin{aligned} b \sin B &= \cos \varphi \, d\omega \\ b \cos B &= d\varphi \end{aligned}\right\} \qquad (330)$$

$d\varphi$ and $d\omega$ are here the differences of the angle, and are also the quantities $d\varphi$ and $d\omega$, already computed. The latter formulæ are used in navigation, substituting, however, the mean of the two latitudes φ and φ', so that the formulæ may be written (Art. 127)

$$\left.\begin{aligned} b \sin B &= \cos \tfrac{1}{2} \Sigma\varphi \, d\omega \\ b \cos B &= d\varphi \end{aligned}\right\} \qquad (331)$$

With these latter formulæ compute the two halves of each conjugate diameter, a' and b', then the proof of the ellipse is given by the equality subsisting between the sum of the squares upon the semiaxes equaling the sum of the squares upon the semiconjugate diameter

$$a^2 + b^2 = a'^2 + b'^2 \qquad (332)$$

And here it may be noted that mathematical accuracy in this equality must not be expected in the numerical values for the shadow of an eclipse, because the shadow is not *rigorously* an ellipse, as already stated, though near enough for practical purposes and with approximate formulæ. It will be noticed that the two halves of a conjugate diameter in this problem on the earth's surface are not of equal length because the degrees of latitude and longitude are not of the same length in different latitudes. To form the limiting curves, we lay off from the centre line certain distances in latitude, then in longitude, laying off equal degrees and minutes for each limit; but toward the equator a degree of longitude contains

more miles than a degree toward the poles, so that these distances *in miles* are greater toward the equator than those toward the poles. The limiting points resulting are therefore *not* at equal distances from the centre line. This line from the centre to the limits is one of our conjugate diameters, and the two halves are seen to be not of equal lengths; and moreover the three points do not lie exactly in a straight line. These circumstances show themselves clearly in the computation following. The effect of this enters into our calculations through the cosine of the latitude in equation (330) given above, and in others of a similar form.

Besides this, there is another cause wholly distinct and acting in different directions, which produces still greater effect upon the length of all lines, and that is the variable effect of the sun's zenith distances at the two ends of any line. The more obliquely the shadow falls upon the earth, the greater its length. It is therefore greatest at the beginning and ending of an eclipse. This effect enters into the calculation of the limiting curves through $\cos\beta$, the zenith distance for a given place, and thence to λ. $\text{Cos}\,\beta$ is here required, but is unknown, and its value is assumed to be the same as for the central line, and $\cos\beta$ is therefore taken from that computation. CHAUVENET states that the limiting curves are not rigorously exact, and this is his principal approximation. We may see the amount of this approximation roughly in figures thus: In the previous computation for $9^h\ 30^m$, $\cos\beta$ in the central line is 9.9449. The width of the shadow path is about two degrees at this time; a change of one degree on each side will change $\log\cos\beta$ by 0.0040, and this will change λ in the same computation by about $0'.55$, making the semi-major axis toward the equator less and the other half toward the pole greater by an amount a little more. The total length would not be much changed, but the centre would not lie in the middle of the axis. These are some of the difficulties we have to contend against in regarding this shadow as an ellipse and the earth a plane.

The numerical work for formulæ (331) and (332) is as follows: We compute each half of the conjugate diameters because they are not of the same length.

The data in the work below are taken from the previous example. Signs are omitted, since it is only numerical values that are required, and the four extremities of the conjugate diameters are noted by the points of the compass. $\frac{1}{2}\Sigma\varphi$ in the formula is the mean of the latitudes of the centre of the eclipse and the extremity of the diameter. The adjacent semidiameters are combined in the lower part of the

example. The errors seem large, but when we compare the sum of the squares of the *whole* diameters—that is, by adding the two partial sums for the half diameters (which can be done in two ways, as shown below)—and then compare with double the sums on the half axes, the check becomes almost rigorously exact.

Proof of the Ellipse.

Data from the Previous Example at $9^h\ 30^m$.

		N. Limit.		S. Limit.		E. Preceding.		W. Following.	
(331)	$d\omega$	$-28'.618$	-1.4566	$28'.618$	1.4567	$61'.86$	1.7914	$61'.86$	1.7914
	$\cos \frac{1}{2}\Sigma\phi$	11° 28.0	9.9912	12° 22.6	9.9898	12° 12.2	9.9901	11° 38.4	9.9910
			1.4478		1.4465		1.7815		1.7824
	$d\phi$	54.59	$+1.7371$	54.59	1.7371	33.75	1.5283	33.75	1.5283
	$\tan B'$	$-27° 11'$	-9.7107		9.7094		0.2532		0.2541
	$\sin \cos B'$		9.9492		9.9494		9.9411		9.9413
	$b \log b'$	$61'.36$	1.7879	$61'.33$	1.7877	$69'.25$	1.8404	$69'.36$	1.8411
	log squares		3.5758		3.5754		3.6808		3.6822

Sum of the Squares on the Axes.

(332)	a	69.45		1.8417
	a^2		$4823'.9$	3.6834
	b	61.18		1.7866
	b^2		$3742'.8$	3.5732
	Sum of squares		8566 .7	

Squares on Conjugate Diameters.

			Sums.	Errors.
	N. $3765'.3$	E. $4795'.1$	$8560'.4$	-6.3
	W. 4810 .8	S. 3761 .8	8572 .6	$+5.9$
Sums	8576 .1	8556 .9	17133 .0	
Errors	$+9$ .4	-9 .8		
Double the squares on the axes			$17133'.4$	
Error			-0 .4	

The reason why the half sums do not check correctly is evidently because in the example we took the two halves exactly as they resulted in the previous example, and the centre was not in the middle of the lines. If we take the *means* of a' and b', the check is rigorous.

The largest error on the half axes, namely, $-9'.8$, represents an error of less than $0'.04$ in a' and b', for if they are each increased by this amount, it is more than sufficient to reduce this error. Therefore, we may regard all the lines correct within about $0'.04$. The axes and their squares are here regarded as being rigorously correct.

174. *Construction of the Ellipse Graphically.*—For most purposes it may be sufficient merely to make a drawing of the ellipse of shadow from the axes computed by the formulæ (Art. 171), or furnished by the *Nautical Almanac* office. If intended for use along an extended stretch of country, it may be necessary to make several drawings of the ellipse, since both the eccentricity and direction of

the major axis usually change greatly during one eclipse. The eccentricity will change the greatest at the beginning and ending upon the earth, while the directions of the major axis will change the more rapidly the path passes near the point Z of the earth (Art. 38). For this purpose the azimuth and zenith distance of the sun are required at the place of the centre of the shadow and at the given time, and also the axes $a = \lambda$ and L, from which b is obtained.

In the following example we will compute these quantities for the ellipse at the beginning and ending of the eclipse given in the example (Section XVIII.). For purposes required in future articles, the ellipses are here computed for the exact times of beginning and ending. Generally, however, if the choice is given, it will be simpler to centre the ellipses upon one of the five-minute points, for which the eclipse data are given in the *Nautical Almanac*. Four-place logarithms for the azimuth will be sufficient, as it is required only to the nearest minute of arc. The quantities φ and ω are points of the centre line of the eclipse interpolated from the *Nautical Almanac* eclipse tables, for the times of beginning and ending, found in Section XVIII.; the other quantities, marked *N. A.*, are from the eclipse tables in the *Nautical Almanac*. The quantities λ and L are from the computing sheets of the eclipse. These, however, for 9^h 30^m have been computed in the foregoing pages of this section.

Computation for the Elements of the Ellipse of Shadow.

For the Beginning and Ending at the Given Place (Section XVIII.).

Data.	T.	$9^h 28^m.56$	$9^h 34^m.04$		$9^h 28^m.56$	$9^h 34^m.04$
N. A.	ϕ	$-11° 38'.4$	$-12° 43'.2$	(321) $\cos\phi \sin d$	$+8.9515$	$+8.9496$
N. A.	ω	121 7 .2	119 8 .2	$\sin\phi \cos d \cos\vartheta$	-9.2711	-9.2980
N. A.	μ_1	142 50 .2	144 12 .4	$\sin\zeta \cos A$	-9.4411	-9.4589
	ϑ	+21 43 .0	+25 4 .2	$\sin\zeta \sin A$	$+9.5664$	$+9.6253$
				A	+126 51	+124 77
	$\sin\vartheta$	$+9.5682$	$+9.6271$			
	$\cos\vartheta$	$+9.9680$	$+9.9570$	(321) $\sin\zeta$	$+9.6632$	$+9.7082$
N. A.	$\sin d$	$+8.9605$	$+8.9604$	$\sin\phi \sin d$	$+8.2654$	$+8.3032$
N. A.	$\cos d$	$+9.9982$	$+9.9982$	$\cos\phi \cos d \cos\vartheta$	$+9.9572$	$+9.9444$
	$\sin\phi$	-9.3049	-9.3428	$\cos\zeta$	9.9483	9.9343
	$\cos\phi$	$+9.9910$	$+9.9892$	ζ	$+27° 25'$	$30° 43'$

Data for the Axes of Ellipse from *Nautical Almanac* Computations.

Time.	$\log \lambda$.		a Numbers.		$\log L$.	$\log (L : \sin 1')$.		b Numbers.	
9^h 20	1.8259	+ 70	66.97	+ 109	8.2551	1.7914	− 22	61.86	− 31
25	.8329	88	68.06	139		.7892	26	61.55	37
30	.8417	105	69.45	169	8.2503	.7866	29	61.18	42
35	.8522	+ 119	71.14	+ 199		.7837	− 34	60.76	− 46
40	1.8641		73.13		8.2440	1.7803		60.30	

(326) (327) **Axes for the Times of Beginning and Ending.**

		$\log \lambda$.	$\log a^2$.	a.	$\log L : \sin 1'$.	$\log b^2$.	b.
Beginning	$9^h\ 28^m.56$	1.8392	3.6784	68′.06	1.7873	3.5746	61′.28
Ending	9 34 .04	1.8501	3.7002	70 .81	1.7843	3.5686	60 .86

From the above data the ellipse can be constructed. The azimuth gives the direction of the major axis from the north point, while a and b are the axes. The best method for construction is that by concentric circles on the axes described in Art. 162. In the present example, the centres must be located on the centre line of the eclipse by their latitudes and longitudes. This is shown in Fig. 23, Plate IX., drawn on a large scale of 20 minutes of arc to one inch and reduced one-half; that is, 40 minutes of arc to an inch in the printed plate. The convergence of the meridian shows itself in this figure, which is taken into account by shortening the degrees of longitude for the latitude of the given place 11° 54,′ making one degree of longitude equal to 58′.69 of latitude and drawing the meridian lines parallel. This avoids *curved* lines for the meridians, but it also occasions a small discrepancy of measurement throughout the drawing.

In fact, in whatever way the earth's curved surface may be represented in a drawing, discrepancies arise. Various expedients have been devised to obviate them, giving rise to the several projections—stereographic, globular, mercators, polyconic, etc. Each has its advantages for the purpose intended, but likewise each has its defects.

The curvature of the centre line is apparent in Fig. 23, but it has been drawn as a straight line between the five-minute points, which are marked by small circles. It is thus seen that we have mechanical as well as mathematical difficulties confronting us in this section.

175. *Computation for the Tangent to the Ellipse at any Point.*—Besides the graphic method, we will now derive a series of formulæ by which the tangent to the ellipse at any point may be computed. The given point must of course lie on the curve of the ellipse, and its latitude and longitude must be accurately known. Then the following data must be taken from the eclipse tables in the *Nautical Almanac*, viz.: The latitudes and longitudes of several points in the centre line, from which to interpolate the centres of the ellipses; sin d, cos d, and μ_1, for computing the azimuth (Art. 171); the logarithms of the axes of the ellipse furnished by the *Nautical Almanac* office or computed by Art. 171.

Points whose latitudes and longitudes are accurately known are marked on Plate IX., Fig. 23, by small circles; the centres of the ellipses are accurately known by interpolation. Between any three of these adjacent points the surface of the earth is considered as a plane triangle passing through the points, and we may first show what error is made in this assumption. The longest line in the following computation is the semi-axis of the ellipses. Assuming a mean value of the two, we have 70′.00 in round numbers. The chord of this arc is twice the sine of half the angle, or in natural numbers 0.0203618. Dividing this by sin 1′, the result for the chord is 69.983, an error of 0′.017 between this and the arc. The rise of the arc above the middle of the chord is 1 — cos of half the angle, or 0.0000518 in parts of radius, and dividing by sin 1′, we have 0′.178. The former is almost insignificant, affecting only the second decimal, while the latter has but little effect upon the lengths of lines, and is the elevation only of a high hill and less than the deviations of the elevations in a mountainous district.

In Fig. 23 are shown the centre line and limits of the total Eclipse of Sept. 9, 1904, and the ellipses of shadow at the instant of beginning and ending of the total phase at the given place P assumed in Section XVIII.

Angles are here measured from the north of the meridian of the centres toward the left as positive; and although the ellipse itself moves from left to right across the position of the given place, the apparent motion of the given place, analytically, is from right to left through the ellipse, which is the positive direction according to this system.

In order that the angle of the path with the meridian may be measured accordingly, it is necessary to change the signs of both $d\varphi$ and $d\omega$, so that in the next following formula the second members are given the negative sign; the angle, therefore, differs 180° from that used in prediction; the two angles are of the same nature, but cannot be directly compared, since one is measured from the axes of coördinates and the other from a meridian. We, therefore, have—

Angle of the centre line of the path of shadow,

$$\left.\begin{aligned} n \sin N &= -\,d\omega' \cos \tfrac{1}{2}\Sigma\varphi \\ n \cos N &= -\,d\varphi \end{aligned}\right\} \qquad (333)$$

$d\varphi$ and $d\omega$ are here interpolated from the differences of the centre line for the times of beginning and ending of the eclipse. For example, 1° 44′.3 is regarded as the motion of longitude for the middle

interval between $9^h\ 25^m$ and $9^h\ 30^m$—that is, at $9^h\ 27^m.5$. The time of beginning is at $9^h\ 28^m.56$, which is $1^m.06$ later, and this is a fraction of 5 minutes (the interval for which ω is given). Therefore, the factor for interpolating is $+\ 0.212$; and this, multiplied by 5.3, the mean of the second differences of the motion, gives $+\ 1.10$, which added to $104^m.3$ gives $105^m.40$, the required value of the motion at the exact time of beginning. $\frac{1}{2}\Sigma\varphi$ is to be understood as the mean of the two latitudes at the five-minute points. This gives the correct angle of the path at the time of beginning, for the curvature of the path shows itself in this figure, as well as in the example following.

Angle between the path and Axis of the Ellipse:

The azimuth being always measured from the south end of a meridian, the angle from the north will be $180 - A$; and the angle e between the path and transverse axis is given by the expression

$$e = N - (180 - A) \tag{334}$$

which is taken as positive in the position of Fig. 23.

Centres of the Ellipse:

$$\left.\begin{array}{l}\text{Given by their latitudes and longitudes by interpolating the}\\ \quad\text{central line for the given times.}\end{array}\right\} \tag{335}$$

We know that these must be the centres of the shadow when the ellipse touches and leaves the given place. In the computation following the latitude and longitude of the given place are placed *between* those of the centres of the ellipse for beginning and ending, from which the correct signs of the differences can be gotten for the next equation.

Find the distances r of the given place from the two centres for beginning and ending by the formula:

$$\left.\begin{array}{l}\text{Regard the signs by taking in all cases for } \Delta\varphi \text{ and } \Delta\omega\text{, the}\\ \quad\text{signs resulting from \textit{given place minus the centre.}}\end{array}\right\} \tag{336}$$

$$\left.\begin{array}{l} r \sin R = \Delta\omega \cos \frac{1}{2}\Sigma\varphi \\ r \cos R = \Delta\varphi \end{array}\right\} \tag{337}$$

r is the required distance and R the angle which this line makes with the meridian, which will result as negative for beginning and positive for ending.

Call υ the angle which this line r makes with the transverse axis of the ellipse, measured from the west end of the axis. It will be obtuse for beginning and acute for ending.

With υ and r it is seen that the given place is now referred to the centre and axes of the ellipse by polar coördinates; and we have the

following relations, taking υ as obtuse for beginning and acute for ending.

$$\upsilon = N - R - e. \tag{338}$$

With this, working backward from the usual formulæ for the angle of the vertical, we get the angle w of the normal to the ellipse at the point P, which is usually in the same quadrant with υ.

$$\tan w = \frac{a^2}{b^2} \tan \upsilon, \tag{339}$$

a and b being the axes of the ellipse previously found. And the angle which the normal makes with the path of the shadow is

$$p = w + e \tag{340}$$

The angle between the tangent and the path is evidently the complement of this angle (since the angle at P is a right angle), but measured in the position of this section, it becomes

$$\text{Angle between the tangent and path} = p - 90^\circ$$

Finally, the angle between the tangent and the meridian from the north point toward the left is

$$\left.\begin{array}{ll} \text{For beginning,} & t = (p - 90) - N \\ \text{For ending,} & t = (p - 90) + (180 - N) \end{array}\right\} \tag{341}$$

These latter may be simplified, but they are convenient as they stand.

Combining the last three equations, we have generally for both beginning and ending the angle of the tangent line with the meridian of the centre

$$t = w + e \mp 90 - N \tag{342}$$

The formulæ are sufficiently simple, and they are illustrated in the following example, which doubtless requires no further explanation. The given place is that of Section XVIII., and other data are from the previous examples of this section. The several lines and angles here computed can be measured in Fig. 23.

Computation for the Tangent to the Ellipse.

Data from the *Nautical Almanac* and Previous Examples.

N. A.	Central line	$9^h\ 20^m$	ϕ −10° 0′.2	−57.0		ω 124° 0′.3	−1 39.4	
		25	10 57 .2	58.1	−1.1	122 20 .9	1 44.3	−4.9
		30	11 55 .3	59.3	1.2	120 36 .6	1 50.0	5.7
		35	12 54 .6	−60.8	−1.5	118 46 .6	−1 57.3	−7.3
		40	−13 55 .4			116 49 .3		

(Art. 151) The times	Beginning	$9^h\ 28^m.56$	Ending	$9^h\ 34^m.04$	
(Art. 174) Azimuth		126° 51′		124° 17′	
(Art. 174) Zenith distance and cosine	27° 25′	9.9483	30° 43′	9.9343	

Angle of the Centre Line of the Path.

(333)	(Change signs)	$d\omega$	+105.42	+9.0229	+112.00	+2.0492
		$\cos(1:2)\Sigma\phi$	−11 26.3	+9.9913	−12 25.0	+9.9897
	$n \sin N$			+2.0142		+2.0389
	$n \cos N$	$d\phi$	58.34	+1.7660	+59.72	+1.7761
		$\tan N$		0.2482		0.2628
	N			+60° 33′		+61° 22′
		$\sin N$		9.9399		9.9433
	$n \log n$		118.66	2.0743	124.62	2.0956

Angle between the Path and Major Axis.

(Art. 174)	$(180 - A)$			+53 9		+55 43
(334)	$e = N - (180 - A)$			+ 7 24		+ 5 39

Centres of the Ellipses.

	Factors for interpolation		+0.712	+9.8525	+0.808	+9.9074
		$\Delta\phi$	−58.1	−1.7642	−59.3	−1.7731
			−41.37	−1.6167	−47.92	−1.6805
		Correction for Δ_2	+0.12		+0.10	
	ϕ		−11 38.45		−12 43.12	
(335)		$\Delta\omega$	−104.3	−2.0183	−110.0	−2.0414
			−1 14.27	−1.8700	−1 28.88	−1.9488
		Cor. for Δ_2	+0.54		+0.50	
			+121 7.17		119 8.22	

	ϕ		ω	
Centre of ellipses at beginning	−11 38.4		+121 7.2	
		15.6		67.2
Position of the given place	−11 54.0		+120 0.0	
		49.1		51.8
Centres of ellipses at ending	−12 43.1		+119 8.2	

Distances of Given Place from Centres of Ellipses.

(336)	Place—Centre	$d\omega$	−67.2	−1.8274	+51.8	+1.7143
(337)		$\cos(1:2)\Sigma\phi$	−11 46.2	+9.9908	−12 18.6	9.9899
		$r \sin R$		1.8182		1.7042
(336)	Place—Centre	$r \cos R$	$\Delta\phi$ −15.6	−1.1931	49.1	+1.6911
(337)		$\tan R$		0.6251		0.0131
		R		−103 20		+45 52
		$\sin R$		9.9881		9.8560
		$r \log r$	67.62	1.8301	70.50	1.8482

Given Place referred to Centres and Axes of Ellipses.

(338)	N	+60° 33′	+61° 22′
	R	−103 20	+45 52
	$N - R$	+163 53	+15 30
	e	+7 24	+5 39
	$v = N - R - e$	+156 29	+9 51

The Normal to the Ellipse.

		$\tan v$	+9.6386	+9.2396
(339)	(Art. 174, Data)	$\log (a^2 : b^2)$	+0.1038	+0.1316
		$\tan w$	+9.7424	+9.3712
	Angle between the normal and axis w		+151° 4′	+13° 14′

Angle of the Tangent with the Meridian of the Centres.

(340)	e	$+7°\ 24'$		$+5°\ 39'$
	$w+e$	$+158\ \ 28$		$+18\ \ 53$
(342)	Constant	$-90\ \ 0$		$-90\ \ 0$
		$+68\ \ 28$		$-71\ \ 7$
	$-N$	$-60\ \ 33$	$180-N$	$+118\ \ 38$
	Angle of the tangent t	$+7\ \ 55$		$+47\ \ 31$

176. *Check upon the Foregoing Formulæ.*—This consists in comparing the length of the line above noted as r on the earth's surface with that called L on the plane of the place P, Section XVIII., on the prediction; this is done by projecting the former line on the plane of the place. They are the same, being the diameter of the shadow.

The lines of greatest and least curvature of a surface, as we read in the calculus, lie at right angles to one another. Likewise in an inclined plane the lines of greatest and least declivity lie at right angles; the latter is evidently a horizontal line of the plane, while the former is made use of in the special method of projection employed by military engineers to determine the positions of planes. The centres of the ellipse and the position of the given place, P, in Fig. 23, are perpendicularly over these same quantities on the plane of the place used in prediction. Fig. 21, Plate VIII., illustrates this. The major axes of the ellipses are lines of greatest declivity in their planes, but the lines r are not, and we therefore do not know the inclination of this latter line to the fundamental plane; but by means of the line of greatest declivity, we can project it thus (Fig. 23): The projection of r on the line of greatest declivity, the major axis is $r \cos \upsilon$; the projection of this on the plane of the place is $r \cos \upsilon \cos \zeta$, ζ being the inclination of this line to the fundamental plane as well as the zenith distances of the sun. Now project this back upon the line L through the angle υ_1, and as $r \cos \upsilon$ is the base of a right-angled triangle, of which r is the hypothenuse, so on the fundamental plane $r \cos \upsilon \cos \zeta$ is also the base of a triangle, of which the hypothenuse is

$$r' = \frac{r \cos \upsilon \cos \zeta}{\cos \upsilon_1} = r \cos \upsilon \cos \zeta \sec \upsilon_1 \qquad (343)$$

But υ_1 is not known, and to find it we must project the angle υ in this manner:

$$\tan \upsilon = \frac{\sin \upsilon}{\cos \upsilon}$$

in which $\cos \upsilon$ is a line of greatest declivity, and its projection on the

lower plane is $\cos \upsilon \cos \zeta$; but $\sin \upsilon$, being a horizontal of the plane, projects into its own length unchanged, and the projection of this angle υ on the lower plane will then be

$$\tan \upsilon_1 = \frac{\sin \upsilon}{\cos \upsilon \cos \zeta} = \frac{\tan \upsilon}{\cos \zeta} \tag{344}$$

These formulæ may be used as they stand, or combining them, we have

$$r_1 = \frac{r \sin \upsilon}{\sin \upsilon_1} \tag{345}$$

In the example below, the data, taken from the previous section, are computed for the times 9^h 28.56 and 9^h 34.04, whereas L in prediction, Art. 151, is computed for $T_0 = 9^h\ 31^m.0$, we must therefore interpolate r_1 for the interval to T_0, which is $+0.445$, and then compare.

L, however, is given in prediction in parts of radius; reduced to minutes we have as the quantity for comparison in minutes

$$\frac{L}{\sin 1'} \tag{346}$$

In the example below the comparison results very closely, verifying not only the numerical computations, but also the accuracy of the formulæ. For $\tan \upsilon$, $\cos \zeta$, $\sin \upsilon$, r, etc., see examples, Arts. 174, 175.

Example, Check upon the Ellipse of Shadow.

Art. 175		9^h 28.56		$9^h\ 34^m.04$	Art. 151. T_0	$9^h\ 31^m.00$
(344) $\tan \upsilon$	156° 29′	9.6386	9° 51′	9.2396		
$\cos \zeta$		9.9483		9.9343	(346) $\log L$	8.2498
$\tan \upsilon_1$		9.6903		9.3053	$\sin 1'$	6.4637
					in arc	1.7861
(345) r	67′.62	1.8301	70′.50	1.8482		
$\sin \upsilon$		9.6009		9.2332		
cosec υ_1		0.3565		0.7034		
	61.31	1.7875	60.93	1.7848		
Difference			— 0.38			
Interpolation 0.445 × 0.38 — .17						
r_1 at T_0		61.14			L in arc	61′.11
			Error 0′.03			

177. *Width of the Shadow Path.*—This, so far as the author is aware, has not before been made a subject for computation; it is however very easily obtained from data on the foregoing pages. If the line between any one of the computed points of the central line and the limiting point be projected upon the perpendicular to the

path, it will give the width. This line is already computed in Art. 172, and is one of the conjugate diameters b' of the ellipse. The normal to the path makes the angle $N \pm 90°$ with the meridian, while the angle of this line is B already computed, whence the angle between the line and the normal is $(N \pm 90°) \sim B$. Signs may be disregarded, since it is only the numerical values we require, and the accent may also be omitted, so that $b \sin (N - B)$ gives the half width of the shadow path in minutes of arc.

According to Col. A. R. Clarke the length of one degree of the meridian is as follows: *

At the equator		110 567.2 metres.
At 30° of latitude		110 848.5 "
At 60° " "		111 414.5 "
One metre	=	3 280 869.33 feet.

This must be divided by 1609.35, the number of metres in one statute mile; the value of b must be divided by 60 to reduce it to degrees, and the result must be doubled to give the whole width of the shadow path. Collecting these constants and giving them by their logarithms we have the formulæ:

Angle of the shadow path with the meridian, computing between two conservative points of the central line (333).

$$\left.\begin{aligned} n \sin N &= \Delta\omega \cos \tfrac{1}{2}\Sigma\varphi \\ n \cos N &= \Delta\varphi \end{aligned}\right\} \tag{347}$$

Length of the line b between the centre and either limit.

$$\left.\begin{aligned} b \sin B &= \Delta\omega \cos \tfrac{1}{2}\Sigma\varphi \\ b \cos B &= \Delta\varphi \end{aligned}\right\} \tag{348}$$

Width of the shadow path in statute miles.

$$\text{At the equator } W = [0.35985]\, b \sin (N \sim B) \tag{349}$$

Constant for 30° [0.36096]; — for 60° [0.36317].

178. *Velocity of the Shadow.*—The quantity n in formula (347), or 333, is the space passed over by the shadow in five minutes, that being the interval for which the points of the centre line and limits are given in the *Nautical Almanac*. To reduce this from minutes, in which the formulæ give it, to statute miles, we require the same constants as used for the width of the shadow path, except the multiplication by 2. Hence we have velocity of the shadow in five minutes, in statute miles.

* *Tables for Polyconic Projection based upon Clarke's Reference Spheroid of* 1866. Special Publication No. 5 of the *U. S. Coast Survey*, 1900.

At the equator $V = [0.05882]\,n$ (350)

Constant for 30° of latitude [0.05993]; — for 60° [0.06214].

179. Computing these two quantities for $9^h\ 30^m$, for which we have already computed the distance b', in Art. 172, we interpolate N and log n from Art. 175 for this time, and have as follows:

WIDTH OF THE SHADOW PATH AND VELOCITY OF THE SHADOW.

Art. 173	Example	At $9^h\ 30^m$	B' $-27°\ 11'$	log b'	1.7879
" 175	"	Interpolated	N $+60\ 46$	log n	2.0817

(349) Equation	$.N \sim B$	87° 57′	(350) log n		2.0817
	sin $(N \sim B)$	9.9997	Constant		0.0592
	log b'	1.7879	Velocity in 5^m	138.3	2.1409
Constant interpolated		0.3602			
Width in miles 140.5		2.1478	Velocity per hour, 1659.6 miles.		

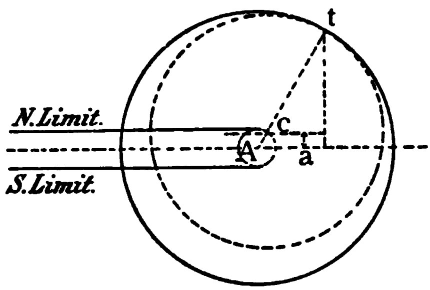
180. *Rigorous Computation of the Ellipse of Shadow.*—Professor CHAUVENET has pointed out in Art. 320 of his chapter on Eclipses that the limits of total or annular eclipse may be computed rigorously by using the formulæ for outline, substituting l_1 for umbra instead of l for penumbra. The ellipse itself as well as the limits may be constructed in this manner entirely rigorously, but the repetitions and approximations would render the computation very laborious.

The expedient resorted to in Art. 172 of this section suggests a method of computing any number of points of an ellipse with comparatively little labor; and the result, while not rigorously exact, will be of the same degree of accuracy as the limiting curves now printed in the *Nautical Almanac*. In Art. 172 we changed the value of Q 90° to obtain the second conjugate diameter. We may likewise compute the whole ellipse round the central point, by assigning to Q successive values as close together as may be necessary in the general formulæ of Section XVI. A great advantage of this method is that the quantities ϑ, L, tan φ_1, φ, ω, μ_1, etc., from the central curve and duration, are constant for all the points of the ellipse.

181. *Circular Shadow.*—If the shadow *of a total eclipse* can be regarded as a circle, the preceding problem is very greatly simplified. In Fig. 24 let the larger circle represent the moon and the smaller the sun, as in the method by semidiameters (Fig. 21, Plate VIII.). If the observer is on the northern limit of the path, he would see the limbs tangent in a line perpendicular to the path and on their northern limbs. If on the southern limit, similarly

tangent on their southern limbs. In the figure the totality is commencing. The point of contact t is always in the line joining the centres of the sun and moon. Let tA (Fig. 24) be the moon's radius, R, and tc that of the sun, r. Let φ be the angle which the line tA makes with the centre line, and a, the distance of the observer, c, from the centre line, then from the figure—

FIG. 24.



$$R \sin \varphi = r \sin \varphi + a$$

or

$$\sin \varphi = \frac{a}{R - r} \tag{351}$$

When the observer is on the northern limit, the line tA, as above remarked, is perpendicular to the path, $\sin \varphi = 1$, and

$$a = R - r \tag{352}$$

which is the half width of the path.

If the observer is on the centre line, $\sin \varphi = 0$ and $a = 0$.

The value of a is always known, being the position of the observer; R and r are also known and their difference. If, therefore, a small circle be drawn round the point A with the radius $R - r$ and a line drawn through it at the distance a from the centre line and parallel thereto, the points where this line cuts the circle will give the directions of the two tangents to the points of contact of the shadow at beginning and ending.

By assuming successive values for a, 0.1, 0.2, 0.3, etc., for $\sin \varphi$, these values will be the ratios of the distance of the point a from the centre line; and considering these also as $\sin \varphi$, the angle φ is found as in the subjoined table, and the angle of the tangent will then be the complement of the angle φ.

$\frac{a}{R-r}$	ϕ
0.0	0° 0′
0.1	5 44
0.2	11 32
0.3	17 27
0.4	23 35
0.5	30 0
0.6	36 52
0.7	44 25
0.8	53 8
0.9	64 9
1.0	90 0

$$\text{Angle of the tangent} = 90 - \varphi$$

When the point c is south of the centre line, a is negative and φ likewise becomes a negative angle. For ending, φ will be greater than 90°.

This simple solution occurred to the author in connection with the total eclipse of May 28, 1900.* The elliptical shadow was not worked out until nearly two years later, when the present work was taken up.

The reader may perhaps ask why it is that in Fig. 20, in which the shadows were drawn as *circles*, the figure should give times and duration, etc., so closely to the finally computed values, when we have just shown that these curves should have been ellipses. The reason is that in Fig. 20 we simply took *proportionate* parts, not *absolute* values; and from the relations that a circle bears to an ellipse, the proportions are the same in both. The ellipse on the earth projects into a circle on the fundamental plane, and the proportions of parts and lines are preserved, as we have shown in Art. 176.

SECTION XXI.

THE CONSTANT k IN ECLIPSES, AND OCCULTATIONS AND OTHER DIFFERENCES.

182. THIS constant is the ratio of the moon's radius to that of the equatorial radius of the earth, and this latter being the moon's parallax, the well-known expression results.

$$k = \frac{s}{\pi} \tag{353}$$

Professor CHAUVENET, in his chapter on Eclipses, Art. 293, p. 448, vol. i.,† gives as the value for eclipses BURCKHARDT'S

$$k = 0.27227 \tag{354}$$

To which value CHAUVENET appends a foot-note, as follows: "The value of k here adopted is precisely that which the more recent investigations of OUDEMANS (*Astron. Nach.*, vol. li., p. 30) gives for eclipses of the sun. For occultations, a slightly increased value seems to be required."

In this same chapter, Art. 341, p. 551, in treating of occultations he gives

$$\log k = 9.435000 \quad [\text{In numbers } 0.272271] \tag{355}$$

which is the same as the previous value, but he appends a foot-note as follows:

"According to OUDEMANS (*Astron. Nach.*, vol. li., p. 30), we should use for occultations $k = 0.27264$, or $\log k = 9.435590$, which

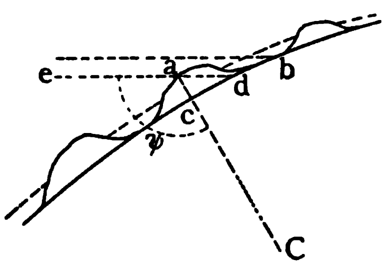
* See *Eclipse Meteorology and Allied Problems*, 1902, by Professor FRANK H. BIGELOW, Department of Agriculture, Weather Bureau Bulletin 267, pp. 55, 56.

† *Manual of Spherical and Practical Astronomy.* Two vols., Lippincott & Co., 1863.

amounts to taking the moon's apparent semidiameter about $1''.25$ greater in occultations than in solar eclipses. But it is only for the reduction of isolated observations that we need an exact value, since, when we have a number of observations, the correction of whatever value of k we may use will be obtained by the solution of our equations of condition."

And here the question naturally arises, Why should there be this difference in this constant? CHAUVENET gives no reason, nor is any generally given in the astronomies, but it is usually understood to be caused partly by irradiation of the sun's light, and partly by the irregularities of the moon's surface. An occultation may occur at the summit of a lunar mountain, as at a, Fig. 25, or in the valley near it, as at b. The recorded times of these two observations would be different, of course, when we might expect them to be nearly alike, resulting in different values from the equations of condition for the semidiameter or for the constant k. And it follows that the constant k can, therefore, be determined more accurately for eclipses than for occultations.

FIG. 25.



It would be difficult to measure the lunar mountains, because it is not their altitude from the moon's surface that is needed, but their height measured on a secant line parallel to the direction of the moon's motions. And, moreover, the libration of the moon will bring other parts of the disk into view at different seasons.

In a total eclipse, as the time of totality approaches, the thin crescent of the sun breaks up into a number of dots or points of light, which have been named "Baily's Beads," from their discoverer. They are now known to be caused by the sun shining between the lunar mountains, the mountains themselves hiding portions of the disk of the sun. It is seen that the time of an occultation on a lunar mountain will be when the beads are found at any one place, and an occultation in a valley will be the time when the beads severally disappear; so that if the time of continuance of these beads be observed at different portions of the moon's disk, the height of the lunar mountains can be computed in the direction of the secant line above alluded to. From a number of observations a mean value may be taken as the moon's semidiameter in occultations. The

dotted curve is supposed to represent the sun's disk broken up into Baily's beads.

If the times of the formation and disappearance of Baily's beads have been observed, and the times predicted accurately, we may find the effect of these two times upon k in the following manner.

The formulæ for the tables and prediction are as follows:

Eq. (35) $$\sin f = \frac{\sin H - k \sin \pi'}{r'g}$$

(36) $$c = z - \frac{k}{\sin f}$$

(38) $$l = c \tan f = z \tan f - \frac{k}{\cos f}$$

(267) $$L = l - \zeta \tan f$$

(268) $$\sin \psi = \frac{m \sin (M - N)}{L}$$

(269) $$\tau = \pm \frac{L \cos \psi}{n} - \frac{m \cos (M - N)}{n}$$

in which k, c, l, L, ψ, and τ are variables, depending upon k primarily and upon one another. In the first of these, a change of k from the value 0.272274 to 0.272509 affects the seventh place of logarithms of $\sin f$ by only nine units of the last place; f is a small angle of about 17′ of arc, and its cosine differing but little from unity that it may be so taken in the following equations. The second equation is contained in the next following.

Differentiating the last four successively and substituting values from previous equations, we have as follows: Taking here the lower sign for total phase

$$dl = -\frac{dk}{\cos f} = -dk$$

$$dL = dl = -dk$$

$$\cos \psi \, d\psi = -\frac{m \sin (M - N)}{L^2} dL = +\frac{L \sin \psi}{L^2} dk$$

$$d\psi = +\frac{\tan \psi}{L} dk$$

$$d\tau = -\frac{\cos \psi \, dL - L \sin \psi \, d\psi}{n}$$

$$= \frac{\cos \psi \, dk}{n} + \frac{\sin \psi \tan \psi \, dk}{n}$$

$$d\tau = \frac{\cos^2 \psi + \sin^2 \psi}{n \cos \psi} dk$$

Therefore, $$dk = n \cos \psi \, d\tau$$

As $d\tau$ may be observed in seconds and decimal, whereas n is given for one minute in the formulæ for prediction, the former quantity must be divided by 60. Hence, we have

$$dk = \frac{n \cos \psi \, d\tau}{60} \tag{356}$$

When the times of the eclipse have been correctly computed beforehand, this formula can be used for giving the difference between the two values of k for eclipses and occultations, $d\tau$ being the interval between the formation of Baily's beads and their disappearance; and dk the corrections for the eclipse value of k, which will always increase the value for occultations. Cos ψ and τ in an eclipse are always both negative for beginning and both positive for ending, so that if taken with these signs, dk will always be a positive quantity, *increasing* the semidiameter of the moon in occultations.

If the times have not been accurately computed for the place of observation, equations of condition must be resorted to for the two times of formation and disappearance of Baily beads.

The suggestion is made that as the lunar mountains are of various heights, the values of dk, resulting from numerous observations, will likewise vary between wide limits, and their adjustment must rest with the judgment of the computer, for these discrepancies are not *errors* to be reconciled. Probably a mean value will be the most generally correct. For this reason also occultations can never be predicted with the accuracy of an eclipse.

OUDEMAN'S value of k, 0.27227 for eclipses,* is derived from the least value of the moon's semidiameter, measured in the *valleys* of the moon. The *mean* value of the semidiameter is 1″.25 greater, giving the constant $k = 0.27264$ for occultations.

J. PETERS † has more recently, from a large number of occultations, deduced the value 0.272518 for occultations, and this value or values, not much different, have recently been adopted in the English and American *Nautical Almanacs* for occultations. For eclipses OUDEMAN'S value, 0.27227 or 272274, has been used in the American *Nautical Almanac* for many years, except the years 1902–3–4. The English *Nautical Almanac* for 1905 has adopted very nearly the same value, by taking the moon's semidiameter 1″.18 less than for occultations.

* *Astron. Nach.*, li., pp. 25–6–30.
† *Ibid.*, cxxxix., Nos. 3296–7.

Equation (356) can easily be shown graphically. In Fig. 25 the height of the lunar mountains, measured in the direction from the centre C, is $c\,a = d\,k$, the increase of the moon's semidiameter. $a\,e$ being the direction of the moon's motion, $C\,a\,e = \psi$ obtuse for the beginning of the eclipse, and $a\,d$, measured along the secant line, is the effect which the lunar mountain produces upon the time of an occultation; and $a\,d$ being the hypothenuse of a right-angled triangle, of which $a\,c$ is one side, we have—

$$a\,d = \frac{d\,k}{\cos\psi}$$

and the time of describing this distance is found by dividing it by n, the space passed over by the moon in one minute, giving $d\,\tau$. Hence,

$$d\,\tau = \frac{d\,k}{n\cos\psi}$$

which is equation (356).

The irradiation of light, as when a bright surface is seen upon a dark one, is also known to increase the apparent size of the bright surface, and thus to cause an apparent diminution of the moon's diameter in solar eclipses. This irradiation is the explanation given to the "Black Drop" in Transits of Venus.*

The black drop can be seen by any one by a simple experiment. When looking at the sky when bright, or the white shade of a gas-lamp, hold the thumb and fingers before one eye, and bring them slowly together; just as they are about to touch, a little swelling will appear on each, that move toward one another. It can sometimes be better seen if the finger is held close to the eye.

183. There is one point in the theory of eclipses and occultations that is rather obscure in the formulæ, and therefore perhaps not generally recognized, which is, that the earth in proportion to the moon is taken *smaller* in eclipses than in occultations. This led me to think that it had some effect upon the constant k, but there is a compensation in the formulæ, so that k is not affected by it. As this investigation reveals some points not otherwise explained, and also shows clearly the differences in the formulæ when used for eclipses and occultations, it may not be amiss to give it a place in this section.

If we wish to ascertain the difference of apparent declination between the sun and moon, we apply their parallaxes, thus:

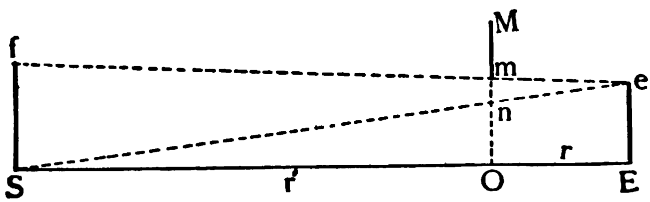
$$(\delta - \pi) - (\delta' - \pi') = \delta - (\pi - \pi') - \delta'.$$

* This is described in Professor NEWCOMB'S *Popular Astronomy*, Ed. 1878, p. 179 *et seq.*

In the second member we have applied the two parallaxes to the moon, while the sun is taken in its true position, being given by its centre. This difference of parallaxes has been styled by Professor LOOMIS the *relative* parallax.

In eclipses the same thing is done, which is illustrated in Fig. 26. Let E be the centre of the earth, S that of the sun, and M the semi-

FIG. 26.



diameter of the moon. The shadow will first touch the earth at e, when the moon's limb is at m on the line ef, drawn tangent to the disks of the sun and moon. But if we attribute the sun's parallax to the moon, the sun becomes analytically a point at S, and the eclipse will commence when the moon thus transformed reaches the point n on the line $e\,S$, drawn to the centre of the sun. We now require to know the length $n\,O$, which is in proportion to $e\,E$ as their distance from S.

$$n\,O : e\,E :: r' - r : r'.$$

Taking for $e\,E$ the parallax π, we have—

$$n\,O = \frac{\pi\,(r' - r)}{r'}$$

But by (23) $r = \dfrac{1}{\sin \pi},$ or nearly $r = \dfrac{1}{\pi}$; and $r' = \dfrac{1}{\pi'}.$

Whence $n\,O = \pi - \pi'.$

This is the radius of the cone of solar parallax at the distance of the moon from the earth, and it represents the earth's radius in BESSEL'S Theory of Eclipses. Professor LOOMIS, in his *Practical Astronomy*,* seems to have recognized the facts here presented, but not clearly. In treating of Eclipses of the Sun, Chapter XI., Art. 249, he says: "The relative parallax is 54′ 19″.1, or 3259″.1, which represents the apparent semidiameter of the earth's disk, if seen at the distance of the moon from the earth." This is merely the definition of the moon's *parallax*. The *relative* parallax cannot be seen, since

* *An Introduction to Practical Astronomy*, by ELIAS LOOMIS, LL.D. Harper and Brother, New York, 7th Edition, 1873.

it is only the radius of the cone of solar parallax at the distance of the moon from the earth.

We will now show this quantity in CHAUVENET'S formulæ. It also occurs in Section XIX. on the Method by Semidiameters. In Art. 113, for central eclipse at noon, we derived the following formula:

$$y = \frac{1}{(1-b)} - \cdot \frac{\delta - \delta'}{\pi}$$

which is rigorous, excepting only that the arc is written for the sine. And this is still further reduced in Art. 159, on the projection by the Method of Semidiameters, by showing that $\pi(1-b) = \pi - \pi'$, or

$$y = \frac{\delta - \delta'}{\pi - \pi'} \tag{357}$$

This formula being computed for the central eclipse at noon, y lies in the axis of coördinates, and can never give real values on the earth if it exceeds unity, therefore, $\pi - \pi'$ must represent the earth, the unit of measure. If the formulæ for x and z be similarly reduced,

$$x = \frac{\cos\delta \sec\delta' (\alpha - a)}{\pi - \pi'}$$

$$z = \frac{\cos(\delta - \delta')}{\pi - \pi'} \tag{358}$$

All of which are fractions having the same denominator, which represents the earth's radius in this theory. At the instant of conjunction, however, $\alpha - a = 0$ and $x = 0$.

The effect of diminishing the denominator of a fraction is to increase its value, and this is the effect produced numerically upon x y and z in eclipses, which can readily be shown, thus: taking the quantity $\alpha - a$, upon which x mainly depends, we know that it is the sum of $(\alpha - \alpha')$, and the small term of equations (17 and 18). What quantity subtracted from π will be the equivalent of this small term?

$$(\alpha - a) : \text{small term} :: \pi : x.$$

Taking $(\alpha - a)$ from the Eclipse Tables, Art. 37, π from the Data, Art. 17, and the logarithm of the small term from the example, Art. 36, all for 12^h, we have—

$$\frac{\alpha - a}{\pi} = \frac{\text{small term}}{x}, \quad \frac{1''.48\ 58''.52}{61'\ 22.600} = \frac{15''.57 \log(1.19139)}{8.75}$$

The sun's parallax for this date is— 8.74

That this result just equals the solar parallax is what we wished to prove. We see now that in order to take for the earth a smaller

value, the coördinates are increased, which has the same effect. In occultations these small terms are equal to zero, making the coördinates smaller than in eclipses, which has the same effect as making the earth larger. The moon's parallax remains the same in both. We thus learn the meaning of these small terms, and thereby arrive at a better understanding of both eclipses and occultations.

In Section XXVI. on Occultations, the formulæ for the coördinates are (Equation 436):

$$x = \frac{\cos\delta \sin(\alpha - \alpha')}{\sin\pi}$$

$$y = \frac{\sin(\delta - \delta')\cos^2\tfrac{1}{2}(\alpha - \alpha') + \sin(\delta + \delta')\sin^2\tfrac{1}{2}(\alpha - \alpha')}{\sin\pi}$$

If these are reduced in the same manner as those for eclipses (357 and 358), we have—

$$x = \frac{\cos\delta\,(\alpha - \alpha')}{\pi}$$

$$y = \frac{\delta - \delta'}{\pi}$$

z is infinity in occultations.

which are similar to the equations for eclipses with the change in the denominators as above noted.

SECTION XXII.

CORRECTIONS FOR REFRACTION AND ALTITUDE.

184. *Correction for Refraction.*—If the sun is low in the horizon during an eclipse, the refraction of the atmosphere will much affect the times. Professor CHAUVENET has discussed this in the following manner in his chapter on Eclipses, Art. 327, p. 515: A ray of light before reaching the earth is refracted toward the normal to the surface, and since the atmosphere increases in density as the ray approaches the earth, its path will be a curve concave toward the earth. This ray, we will suppose, reaches an observer at some point (A) on the earth's surface, and he sees the sun apparently higher in the heavens than it really is. If there were no refraction, the ray of light, if continued in a *straight* line, would reach the vertical line above A at some point (B); at which point, if there were no refraction, an observer would see a true contact at the same instant as the observer at B sees the contact of the refracted ray.

CHAUVENET'S method of taking account of the refraction is to substitute the point B for A in the formulæ for the position of the observer. Let s be the height of B above A and ρ the radius of the earth, as in all the preceding formulæ. It is then only necessary to substitute $\rho + s$ instead of ρ in the formulæ for ξ, η, and ζ, such as equations (131), (132), Art. 81; (261), Art. 150, etc.; or else more simply write $\rho\left(1 + \frac{s}{\rho}\right)$ for ρ. "Therefore," says Professor CHAUVENET in Art. 327, "when we have computed the values of $\log \xi$, $\log \eta$, and $\log \zeta$ by those equations [(483) of his work] in their present form, we shall merely have to correct them by adding to each the value of $\log\left(1 + \frac{s}{\rho}\right)$."

This logarithm he has computed by the following equation, of which he gives the derivation:

$$1 + \frac{s}{\rho} = \frac{\mu_0 \sin Z'}{\sin Z} \tag{359}$$

in which Z is the true zenith distance of the sun or ray of light, and Z' the apparent zenith distance, and μ_0 the index of refraction of the air at the observer.

The following table is given by CHAUVENET, computed for the values of the mean refraction tables given in his work; that is, β and γ of the tables each $= 1$ and a mean value of $\mu_0 (= 1.0002800)$. Log ζ is the argument and $\log \xi$, $\log \eta$, and $\log \zeta$ are each to be corrected by the same amount, taken from the following table, which he has deduced from BESSEL:

CORRECTION FOR REFRACTION FOR LOG ξ, LOG η, AND LOG ζ (CHAUVENET'S *Astron.*, i., Art. 327).

$\log \zeta$.	Correction for logs of ξ, η, ζ.	$\log \zeta$.	Correction for logs of ξ, η, ζ.	$\log \zeta$.	Correction for logs of ξ, η, ζ.
0.0	0.0000000	9.0	0.0000119	8.0	0.0000788
9.9	0001	8.9	0167	7.9	0835
9.8	0002	8.8	0225	7.8	0875
9.7	0005	8.7	0292	7.7	0909
9.6	0008	8.6	0367	7.6	0937
9.5	0.0000014	8.5	0.0000446	7.4	0.0000978
9.4	0023	8.4	0525	7.2	1006
9.3	0035	8.3	0602	7.0	1023
9.2	0054	8.2	0672	6.5	1044
9.1	0081	8.1	0734	6.0	1051
9.0	0.0000119	8.0	0.0000788	$-\infty$	0.0001054

185. *Correction for the Altitude of the Observer above the Sea-level.* —If s' is the altitude of the observer above the sea, it is only necessary to substitute $\rho + s'$ instead of ρ in the general formulæ of an eclipse, which is simply done by adding to $\log \xi$, $\log \eta$, and $\log \zeta$ the value of $\log\left(1 + \frac{s'}{\rho}\right)$. From the theory of logarithms we have generally

$$\log\left(1 + \frac{s'}{\rho}\right) = \log \rho = M\left(\frac{s'}{\rho} + \tfrac{1}{2}\left(\frac{s'}{\rho}\right)^2 +, \text{etc.}\right) \tag{360}$$

in which M is the modulus of the system. The second term is small compared with $\log \rho$, so that all the terms within the parentheses except the first may be omitted; and assuming for $\log \rho$ a mean value for latitude $= 45°$, we have for s', when expressed in English feet,

$$\text{Correction for } \log \xi,\ \log \eta, \text{ and } \log \zeta = 0.000\ 000\ 020\ 79\, s' \tag{361}$$

If s' is expressed in meters, we have for the constant, instead of the above,

$$0.000\ 000\ 064\, s' \tag{362}$$

For example, if the observer is 1000 feet above the sea, the correction is 0.0000208, to be added to $\log \xi$, $\log \eta$, and $\log \zeta$.

SECTION XXIII.

SAFFORD'S TRANSFORMATION OF ECLIPSE FORMULÆ.

186. THESE formulæ were used in the old *Almanacs* down to the year 1881 inclusive, but for the subsequent years BESSEL's notation has been restored. In BESSEL's notation the quantities given in the eclipse tables of the *Nautical Almanac* have an astronomical signification, but by this transformation the quantities on the other hand have not. BESSEL's elegant formulæ, instead of being simplified, are only obscured. The transformation is by Professor TRUMAN HENRY SAFFORD,* who was engaged upon the *Nautical Almanac* during some of its earlier years.

* These are given by CHAUVENET, but had previously been printed by Professor BENJAMIN PEIRCE in his *Spherical Astronomy*.

In the fundamental equation of eclipse

$$(x-\xi)^2 = (l-\zeta\tan f)^2 - (y-\eta)^2 \tag{363}$$

Assume
$$a^2 = b^2 - c^2 \tag{364}$$

Then
$$a = x - \xi \tag{365}$$
$$b = (l-\zeta\tan f) + (y-\eta) \tag{366}$$
$$c = (l-\zeta\tan f) - (y-\eta) \tag{367}$$

We have for the coördinates of the given places

$$\left.\begin{aligned} \xi &= \rho\cos\varphi'\sin\vartheta \\ \eta &= \rho\sin\varphi'\cos d - \rho\cos\varphi'\sin d\cos\vartheta \\ \zeta &= \rho\sin\varphi'\sin d + \rho\cos\varphi'\cos d\cos\vartheta \end{aligned}\right\} \tag{368}$$

which are the principal equations from which the various equations for the coördinates have been derived in the foregoing sections, as, for example, equation (361), Art. 150, in prediction. Substituting these values in the two previous equations, they become

$$\left.\begin{aligned} b = l + y &- \rho\sin\varphi'(\cos d + \sin d\tan f) \\ &+ \rho\cos\varphi'(\sin d - \cos d\tan f)\cos\vartheta \\ c = l - y &+ \rho\sin\varphi'(\cos d - \sin d\tan f) \\ &- \rho\cos\varphi'(\sin d + \cos d\tan f)\cos\vartheta \end{aligned}\right\} \tag{369}$$

Place

$$\left.\begin{aligned} A &= x \\ B &= l + y \\ C &= -l + y \\ E &= \cos d + \sin d\tan f = \cos(d-f)\sec f \\ F &= \cos d - \sin d\tan f = \cos(d+f)\sec f \\ G &= \sin d - \cos d\tan f = \sin(d-f)\sec f \\ H &= \sin d + \cos d\tan f = \sin(d+f)\sec f \end{aligned}\right\} \tag{370}$$

We also have, as in the previous method, μ (μ_1) and

$$\mu' = \frac{\Delta\mu_1\sin 1''}{3600}10^6 \text{ for the meridian of Washington.} \tag{371}$$

All of these eight quantities last given are independent of any place on the earth, and constitute the eclipse tables of this method. The tables also give A', B', C'. The changes of A, B, C, in one second, and also B, C, E, F, G, H for shadow; or as E, F, G, H differ from those quantities for penumbra by a small constant, this constant is given instead of the quantities themselves. It is to be noted also that in the *Nautical Almanac* the quantities A', B', C', and μ' the changes of μ are given in units of the sixth decimal, so that to form them these quantities in the formula of this section, as well as μ' in equation (376) and τ in equation (381), must be multiplied by 10^6.

187. *Prediction by this Method.*—Substituting the values of (370) in (369), we have

$$\left.\begin{aligned} a &= x - \xi = A - \rho \cos \varphi' \sin \vartheta \\ b &= B - E\rho \sin \varphi' + G\rho \cos \varphi' \cos \vartheta \\ c &= -C + F\rho \sin \varphi' - H\rho \cos \varphi' \cos \vartheta \end{aligned}\right\} \tag{372}$$

And the fundamental equation becomes

$$a = \sqrt{bc} \tag{373}$$

For any assumed time T_0, take from the eclipse tables the proper quantities and compute a, b, and c by equation (372), and if a differs from $\sqrt{bc}$, the assumed time requires a correction, to be found as follows:

Place $$m = \sqrt{bc} \tag{374}$$

$a', b', c' =$ the changes of a, b, and c in one second.

$\tau =$ the required correction for the assumed time T_0.

Then $a + a'\tau = m + m'\tau$

Whence $$\tau = \frac{m - a}{a' - m'} \tag{375}$$

To find a', differentiate equation (372), remembering that as $\omega = \mu_1 - \vartheta$, $d\mu_1 = d\vartheta$, and distinguishing the derivatives by accents,

$$a' = A' - \mu'\rho \cos \varphi' \cos \vartheta \times 10^6 \tag{376}$$

And to find m',

$$B' = y' = C'$$
$$b' = B' - \mu' G\rho \cos \varphi' \sin \vartheta$$
$$c' = -C' + \mu' H\rho \cos \varphi' \sin \vartheta$$

Since f is small in these expressions, we may place $G = H$. Hence,

$$b' = -c' = B' - \mu' G\rho \cos \varphi' \sin \vartheta \tag{377}$$

And from (374)

$$2mm' = cb' + bc' = (c - b)b'$$

$$m' = \tfrac{1}{2}\left(\sqrt{\frac{c}{b}} - \sqrt{\frac{b}{c}}\right)b' \tag{378}$$

Assume $$\tan \tfrac{1}{2} Q = \sqrt{\frac{c}{b}} = \frac{c}{m} = \frac{m}{b} \tag{379}$$

$$m' = -b' \cot Q \tag{380}$$

Then τ is found from the following:

$$\tau = \frac{m - a}{a' + b' \cot Q} \cdot 10^6 \tag{381}$$

In applying these formulæ for prediction

Compute a, b, c, by (372).

m by (374) (which usually has the same sign as a).

a' by ((376) and b' by (377).

$\tan \frac{1}{2} Q$ by (379).

and finally τ by 381.

And the correct time is in Washington mean time.

$$T = T_0 + \tau \tag{382}$$

Or in local mean time, ω being the longitude from Washington.

$$T = T_0 + \tau - \omega \tag{383}$$

With this time a second approximation can be made.

The angle Q here is the same quantity as that used throughout the preceding pages, for we have from (380) at the instant of contact.

$$\tan Q = -\frac{b'}{m'} = \frac{2m}{b - c}$$

And from (365–6–7)

$$\tan Q = \frac{x - \xi}{y - \eta} \tag{384}$$

Q has the same sign as a for the penumbra or umbra of an annular eclipse, but has a contrary sign for the shadow of a total eclipse. As before, it is measured from the north point toward the east.

The angle from the vertex is given by the following equations:

$$\begin{aligned} p \sin P &= \sin \varphi \\ p \cos P &= \cos \varphi \cos \vartheta \\ c \sin C &= \cos P \tan \vartheta \\ c \cos C &= \sin (P - \delta') \\ V &= Q - C \end{aligned} \tag{385}$$

δ' here is the sun's declination.

The magnitude of the eclipse is found as follows, being given in the antiquated *digits.*

Place $2\theta =$ The difference of the two values of Q for beginning and ending, having regard to signs.

$$n = \frac{s}{s'} = \frac{\text{semidiameter of the moon}}{\text{semidiameter of the sun}}$$

$$M = \begin{cases} 12\,(1 + n) \sin^2 \frac{1}{2}\,\theta & \text{if } \theta \text{ is acute} \\ 12\,(1 + n) \cos^2 \frac{1}{2}\,\theta & \text{if } \theta \text{ is obtuse} \end{cases} \tag{386}$$

By omitting the constant 12, the magnitude as usually now given will be expressed as a fraction of the sun's diameter.

PART II.

SECTION XXIV.

LUNAR ECLIPSES.

188. *Introduction.—Criterion.*—This subject comprises Arts. 338 and 339 in Professor CHAUVENET's chapter on Eclipses. There is much similarity between the formulæ for solars and lunars, so that reference to the former will often explain a kindred point in this section. Lunar eclipses cannot be observed with the same degree of accuracy as solar—the approach of the eclipse not being so well marked—for which reason they are considered of much less importance, and the formulæ consequently are not so rigorously exact as those in the preceding sections. The earth's atmosphere refracts some of the sun's rays on the boundaries of the cone of shadow, making it larger than the true diameter of the earth would cause. This is estimated as one-sixtieth in LOOMIS' *Astronomy*, and one-fiftieth in CHAUVENET's chapter. As the true amount of this refraction is not known, the error cannot be determined. The spheroidal form of the earth is another source of error, which would necessitate a second approximation; but as this is small, the earth is taken as a sphere of the radius of latitude 45°, which is too small at the equator and too large at the poles. And as the refraction in the first case would cause a gradual diminution of the sun's rays, this is undoubtedly the reason why a lunar eclipse cannot be observed as accurately as a solar eclipse—the moon being *gradually* obscured.

The criterion for lunar eclipses is similar to that of solar (Art. 8), except that here we have

$$\beta \cos I' < \tfrac{51}{50} (\pi - s' + \pi') + s \tag{387}$$

or with a mean value of I'.

$$\beta < [\tfrac{51}{50} (\pi - s' + \pi') + s] \times 1.00472$$

The small term varies from 15″.3 to 18″.0, or with a mean value.

$$\beta < \tfrac{51}{50} (\pi - s' + \pi') + s + 16''.6 \tag{388}$$

With the values adopted in Art. 10, we have

$$\left.\begin{array}{ll} \beta > 63'\,51'' & \text{Eclipse impossible.} \\ \beta < 53'\,29'' & \text{Eclipse certain.} \end{array}\right\} \qquad (389)$$

Between these limits eclipse doubtful.

These values differ somewhat from CHAUVENET'S, but it is believed that they are more correct.

As in solar eclipses, the best way to ascertain what eclipses there will be in a certain year is by means of the Saros (Art. 11); but this method will not show a new series of eclipses entering, for which the above criterion must be employed, or else in close cases proceed as if there were an eclipse and compute it; and if sin ψ is less than unity there will be an eclipse.

The motion of the series of successive eclipses is governed by the moon's node as in solars. At the moon's *ascending* node the series is moving south, and at the descending node it is moving north, the quantity $\delta + \delta'$ at the time of opposition shows the amount of motion of successive eclipses. If the eclipse is known to be north of the equator and moving north, for example, the durations of each successive eclipse are decreasing. If the series is decreasing, as shown by $\delta + \delta'$, the durations and magnitudes will generally also decrease; but these latter are no standard for the general amount of decrease as the quantity $\delta + \delta'$ always is. A curious case occurred in the year 1898, the partial lunar of January 7. The series was moving north, and the successive eclipses decreasing at the north pole. Compared with previous eclipses of the series 1879, December 27, I found that the duration of the shadow had *increased*. Both eclipses of this year exhibited the same peculiarity, which is shown as follows:

Durations.					Magnitude.
	Penumbra.		Shadow.	Totality.	
1843, Dec. 6,	$5^h\ 8^m.2$		$1^h\ 46^m.2$	. . .	0.202
		-3.2			
1861, Dec. 16,	5 5 .0		1 42 .2	. . .	0.185
		-0.3			
1879, Dec. 27,	5 4 .7		1 35 .9	. . .	0.164
		$+7.8$			
1898, Jan. 7,	5 12 .5		1 35 .5	. . .	0.157
1844, Nov. 24,	$6^h\ 15^m.2$		$3^h\ 49^m.8$	$1^h\ 33^m.0$	1.435
		-2.7			
1862, Dec. 5,	6 12 .5		3 49 .7	1 32 .0	1.415
		$+2.4$			
1880, Dec. 15,16,	6 14 .9		3 48 .9	1 29 .9	1.388
		$+2.9$			
1898, Dec. 27,	6 17 .8		3 48 .9	1 29 .2	1.384

The cause of this eccentricity seems to be that while the point given by $(\delta + \delta')$ in the axis of Y properly moved north, the inclination of the path also greatly increased, so that the middle point was brought nearer to the centre of the penumbral shadow. If the inclination had increased a little more, the duration of the shadow of the December eclipse would also have increased; and it seems possible that the next eclipse of the series, 1917, Jan. 8, may have an increased magnitude. A change of constants used would make some differences in the times, but would not likely make the difference of $5^m.3$ shown above. Theoretically, this cause alone would have increased both durations, but doubtless the other changes in the elements partially counteracted its effects; for example, the sun's semidiameter acts upon the shadow and penumbra with contrary signs. The circumstance was not investigated at the time. The total eclipse of Dec. 27, of the same year, 1898, exhibits a similar peculiarity.

189. *General Formulæ.*—The notation here is generally the same as in solar eclipses, but instead of the sun we have for the centre of the earth's shadow,

$\alpha' =$ Sun's R. A. $+ 12^h$ (retained in time).

$\delta =$ Declination of the shadow $= -$ sun's declination.

$L =$ Distance between centres of moon and shadow.

In the spherical triangle formed by the centres and pole

$$\left.\begin{aligned} L \sin Q &= \cos \delta \sin (\alpha - \alpha') \\ L \cos Q &= \cos \delta' \sin \delta + \sin \delta' \cos \delta \cos (\alpha - \alpha') \end{aligned}\right\} \tag{390}$$

As the earth is not a sphere, a mean value of its radius at 45° is substituted for the variable radius in computing the moon's parallax, so that instead of π in the formulæ, we have

$$\pi_1 = [9.99928]\pi \tag{391}$$

And the radii of the shadow and penumbra increased by $\frac{1}{50}$ are

$$\left.\begin{aligned} \text{Radius of shadow} &= \tfrac{51}{50}(\pi_1 - s' + \pi') \\ \text{Radius of penumbra} &= \tfrac{51}{50}(\pi_1 + s' + \pi') \end{aligned}\right\} \tag{392}$$

Therefore, we have for L

$$\left.\begin{aligned} &\text{First and last contacts with penumbra,} && L = \tfrac{51}{50}(\pi_1 + s' + \pi') + s \\ &\text{“ \quad “ \quad “ \quad shadow,} && L = \tfrac{51}{50}(\pi_1 - s' + \pi') + s \\ &\text{Second and third “ \quad “} && L = \tfrac{51}{50}(\pi_1 - s' + \pi') - s \end{aligned}\right\} \tag{393}$$

The second and third contacts with penumbra are not required. They can be found by changing the sign of s.

These formulæ need not be so rigorously exact, as above remarked; we may write the arc for the sine in (390), which gives

$$\left.\begin{aligned} L \sin Q &= (a - a') \cos \delta \\ L \cos Q &= \delta + \delta' - \frac{\sin 2\delta \sin^2 \frac{1}{2}(a - a')}{\sin 1''} \end{aligned}\right\} \tag{394}$$

Place
$$\epsilon = \frac{\sin 2\delta \sin^2 \frac{1}{2}(a - a')}{\sin 1''}$$
$$= [6.4357] \sin 2\delta\, (a - a')^2 \tag{395}$$

Also
$$\left.\begin{aligned} x &= 15\,(a - a') \cos \delta \\ y &= \delta + \delta' - \epsilon \\ x'\, y' &= \text{The hourly changes of } x \text{ and } y \end{aligned}\right\} \tag{396}$$

And (394) becomes

$$\left.\begin{aligned} L \sin Q &= x + x'\,\tau \\ L \cos Q &= y + y'\,\tau \end{aligned}\right\} \tag{397}$$

To solve the above by finding τ, which is analogous to that quantity in solar eclipses, place

$$\left.\begin{aligned} m \sin M &= x_0 \\ m \cos M &= y_0 \end{aligned}\right\} \text{for the epoch hour.} \tag{398}$$

$$\left.\begin{aligned} n \sin N &= x' \\ n \cos N &= y' \end{aligned}\right\} \tag{399}$$

Then
$$\sin \psi = \frac{m \sin (M - N)}{L} \tag{400}$$

$$\tau = \frac{L \cos \psi}{n} - \frac{m \cos (M - N)}{n} \tag{401}$$

$$T = T_0 + \tau \tag{402}$$

Local time for any place, $T = T_0 + \tau - \omega$ (403)

As in solar eclipses, $\cos \psi$ is to be taken with the negative sign for beginning, giving an obtuse angle; and with the positive sign for ending, giving an acute angle.

The time of greatest obscuration, which is usually also considered the middle of the eclipse, is

$$T_1 = T_0 - \frac{m \cos (M - N)}{n} \tag{404}$$

The angle of position of the point of contact on the moon's limb, measured from the north point toward the east,

$$Q = 180 + N + \psi \tag{405}$$

The least distance of the centres

$$\varDelta = \pm m \sin (M - N) \tag{406}$$

the double sign, implying only that $\varDelta$ shall be positive. And the magnitude of the eclipse

$$M = \frac{L - \varDelta}{2s} \tag{407}$$

in which the value of L for total shadow is to be used, and $\varDelta$ interpolated for the middle of the eclipse. For a partial eclipse, M is less than unity, but for a total eclipse it is greater than unity, and shows how far the moon is immersed in the earth's shadow.

Moon in the Zenith.—The formulæ for this problem are not given by Professor CHAUVENET, nor do I remember seeing them in any astronomy. They are, however, easily derived as follows: The Greenwich mean time at any instant gives the longitude of the meridian of the mean sun, T. By adding the equation of time, E (reducing from mean to apparent time), we have the longitude of the meridian of the true sun, $T + E$; the meridian of the centre of the shadow being 12 hours greater, or $T + E + 12^h$. As the moon in the heavens moves *toward* the east, while longitudes are measured *from* the east, the moon's position at the beginning of an eclipse will numerically increase the above quantity, and decrease it for ending. The distance of the moon from the centre of the shadow is $(a - a')$, which is negative for beginning and positive for ending; hence, it must be algebraically subtracted from the above. Hence, we have the moon's longitude in arc at the time, T.

$$\omega = 15\,[T + E + 12^h - (a - a')] \tag{408}$$

$(a - a')$ must be interpolated for the time T; and the factor 15 is merely the reduction from time to arc.

The latitude is simply the moon's declination at the times T.

$$\varphi = \delta \tag{409}$$

These are the positions of the places on the earth which have the moon in the zenith, for the first and last contacts with the moon's limb. They are not used for the other times; and they give the zenith of the earth's hemisphere from which the contacts can be seen.

190. *Data, Elements, Example.*—The data for a lunar eclipse are, with one exception, the same as those for solars (Section III.); the right ascension, declination, semidiameter, and parallax of the sun and moon, interpolated from the *Nautical Almanac*, for at least three

hours before and after the time of opposition in right ascension. But instead of the sidereal time, we here require the equation of time from page II. for the month from the *Nautical Almanac*, and for the preceding and succeeding noons. To find the time of opposition approximately, the most convenient method is by the Saros, Art. 11, in Solar Eclipses; but a new series of eclipses may enter, so that search should be made at suspected places in the Ephemeris, by means of the Criterion.

The right ascension should be interpolated to the decimals of a second, and it is more convenient to retain them in time, though CHAUVENET reduces them to arc—one decimal of a second of arc is sufficient throughout this work.

There being no Lunar Eclipse in the year 1904, from which the example for the solar eclipse in the foregoing pages was taken, an example here is selected from the year 1902. The data above mentioned are not here given separately, but placed in the example below, where they can be more conveniently made use of. Differences also are here omitted for want of space.

TOTAL LUNAR ECLIPSE, 1902, APRIL 22.

EXAMPLE, PRELIMINARY COMPUTATION.

G. M. N.	☉ App. R. A.	☽ App. R. A.	$(\alpha - \alpha')$.	log $(\alpha - \alpha')$.	cos δ.	log x.
3^h	$1^h\ 57^m\ 31^s.96$	$13^h\ 50^m\ 7^s.32$	$-7^m\ 24^s.64$	−2.64801	+9.99074	−3.81484
4	41 .30	52 7 .31	5 33 .99	.52374	050	.69036
5	50 .65	54 7 .36	3 43 .29	.34887	031	.51527
6	59 .99	56 7 .47	1 53 .52	2.05123	010	.21743
T_0 7	58 9 .37	58 7 .64	−0 1 .69	−0.22789	.98988	−1.39386
8	18 .68	14 0 7 .87	+1 49 .19	+2.03819	967	+3.20395
9	28 .03	2 8 .16	3 40 .13	.34268	944	.50821
10	1 58 37 .37	14 4 8 .52	+5 31 .15	2.52002	+9.98922	+3.68533

	x.		x'.	π.		$(\pi_1+\pi')$.	$(s$—Const.)	$(s'$—Const.)
3^h	−6528.8	+1626.9	+1626.0	54 43.59		3287.0	894.8	954″.33
4	4901.9	1626.4	5.7	42.77		6.2	4.5	.32
5	3275.5	1625.7	5.3	41.95	π' +8.75	5.4	4.3	.31
6	1649.8	1625.0	5.0	41.14	cor −5.37	4.6	4.1	.30
T_0 7	− 24.8	1624.2	1624.6	40.33	+3.38	3.8	3.8	.29
8	+1599.4	1623.2	4.2	39.53		3.0	3.6	.28
9	3222.6	+1622.8	3.7	38.74		2.2	3.4	.27
10	+4845.4		+1623.4	54 37.96		3281.4	893.3	954.26

	☉ App. Dec. δ'.	☽ App. Dec. δ.	$(\delta+\delta')$	sin 2δ.	$(\alpha-\alpha')^2$.	log ϵ.	ϵ.		
h	° ′ ″	° ′ ″	′ ″						
3	+12 0 53.92	−11 47 26.8	+13 27.12	−9.6019	+5.2960	−1.3336	−21.56	+9.27	
4	1 44.59	55 29.9	+ 6 14.69	.6066	5.0475	1.0898	12.29	6.74	−2.53
5	2 35.24	12 3 29.6	− 0 54.36	.6110	4.6977	0.7444	5.55	4.12	2.62
6	3 25.86	11 25.9	8 0.04	.6158	4.1024	0.1539	1.43	+1.43	2.69
T_0 7	4 16.47	19 18.7	15 2.23	.6202	0.4557	6.5116	0.00	−1.37	2.80
8	5 7.05	27 8.0	22 0.95	.6243	4.0764	0.1364	1.37	4.25	2 88
9	5 57.61	34 53.7	28 56.09	.6286	4.6854	0.7497	5.62	−7.21	−2.96
10	+12 6 48.15	−12 42 35.8	−35 47.65	−4.6327	+5.0400	−1.1084	−12.83		

	y.		y'.	$\log x'$.	For T_0.		For first approx.	
3^h	$+$ 828.7		-433.0	$+3.21112$	x_0	-1.39386	x'	$+3.21075$
		-441.7						
4	$+$ 387.0		429.7	104	y_0	-2.95530	y'	-2.62377
		435.8						
5	$-$ 48.8		426.7	093	tan M	8.43856	tan N	$+0.58698$
		429.8						
6	478.6		423.6	085	M	$+181$ 34 21	N	$+104$ 30 42
		423.6						
T_0 7	902.2		420.5	075	cos	9.99984	sin	9.98592
		417.4						
8	1319.6		417.4	064	log m	$+2.95546$	log $1:n$	$+6.77517$
		410.9						
9	1730.5		414.1	051				
		-404.3						
10	-2134.8		-410.9	$+3.21043$				

Example, Preliminary Work.—After interpolating a and $a' + 12^h$ get the quantity $(a - a')$ in time, then x by 396. By retaining $(a - a')$ in time, it is reduced to arc by the logarithm of 15 when adding the other logarithms. The hourly variation is similar to that in solar eclipses (Art. 30), the epoch hour T_0 being first assumed near the middle of the eclipse, and the hourly motions of x then computed.

π may be reduced to the latitude of 45° by means of the subjoined table. The correction is always to be deducted numerically, and the simplest method of applying it is to deduct it from the sun's parallax π', which is constant, and apply this to π, giving at once $(\pi_1 + \pi')$.

Reduction of π to 45°.

π.	cor.
52″	$-5''.1$
53	5 .2
54	5 .3
55	5 .4
56	5 .5
57	5 .6
58	5 .7
59	5 .8
60	5 .9
61	6 .0
62	-6 .1

The two semidiameters are to have the constants of irradiation deducted as in solar eclipses, Art. 21.

The small term ε is conveniently computed by formula (395), since log sin 2δ can be gotten mentally to four-place decimals, $(a - a')$ is given above, and simply doubled for the square, and the constant given by its logarithms is composed of the constants in CHAUVENET'S form—viz.,

$$\frac{(\frac{1}{2} \sin 1'' \times 15)^2}{\sin 1''}$$

This term is to be algebraically subtracted from $\delta + \delta'$. Proceed with the formula, getting y, and thus the hourly motions y'. If y changes signs, it indicates a large eclipse. The logarithm of x' may be conveniently gotten, but y' must be retained in numbers, since its logarithm cannot be easily interpolated.

All these quantities should be differenced, but these are omitted in the example for want of space.

The quantities mM, nN, for the epoch hour are next required; the two former are computed but once for the eclipse, but the latter are for the first approximation only (formulæ (398), (399)).

Elements.—These may now be gotten from the preceding data, and are similar in all respects to those for solar eclipses (Art. 21), except

that we have here opposition instead of conjunction; the formulæ for these are the same.

191. *The Times, Angles of Position, Magnitude.*—For the times two approximations are required, taking for the first the several quantities for the epoch hour or an hour near the middle of the eclipse, and computing in three columns instead of six, as in the second approximation; but after the angle ψ is reached, there will then be six columns on account of the two values of $\cos\psi$. This first approximation is omitted in the example, but the resulting times are placed at the head of the columns, for which time the quantities are to be taken from the eclipse data above computed. If a total eclipse of eighteen years ago has become partial, it will be found that

COMPUTATION FOR THE TIMES.

	Penumbra.		Shadow.		Totality.	
T	$3^h\ 48^m.95$	$9^h\ 56^m.60$	$5^h\ 0^m.20$	$8^h\ 45^m.35$	$6^h\ 10^m.09$	$7^h\ 35^m.47$
(393) $\pi_1 + \pi'$	3286.4	3281.4	3285.4	3282.4	3284.5	3283.3
s'	+954.3	+954.3	—954.3	—954.3	—954.3	—954.3
Σ	4240.7	4235.7	2331.1	2328.1	2330.2	2329.0
1 : 50	84.8	84.7	46.6	46.6	46.6	46.6
$\frac{51}{50}(\pi_1 + \pi \pm s')$	4325.5	4320.4	2377.7	2374.7	2376.8	2375.6
s	+894.6	+893.2	+894.3	+893.4	—894.1	—893.7
L	5220.1	5213.6	3272.0	3268.1	1482.7	1481.9
(399) x_0'	+3.21106	21043	21093	21054	21083	+3.21068
y_0'	—2.63377	61384	63012	61794	62644	—2.62180
tan N	0.57729	59659	58081	59260	58439	0.58888
N	+104 49 29	104 12 25	104 42 36	104 19 58	104 35 40	+104 27 3
sin	9.98530	98651	98552	98627	98575	9.98604
log 1 : n	+6.77424	77608	77459	77573	78492	+6.77536
(400) $M-N$	+76 44 52	77 21 56	76 51 45	77 14 23	76 58 41	+77 7 18
sin $(M-N)$	+9.98828	98936	98848	98914	98868	+9.98894
cos $(M-N)$	+9.36029	33991	35657	34415	35280	+9.34807
m sin $(M-N)$	+2.94374	94482	94394	94460	94414	+9.94440
log L	+3.71768	71714	51481	51429	17105	+9.17082
sin ψ	+9.22606	22768	42913	43031	77309	+9.77358
ψ			+164 25	+15 38		
cos ψ	—9.99377	+9.99372	—9.98373	+9.98364	—9.90589	+9.90562
(401) log (1)	—0.48569	+0.48694	—0.27313	+0.27366	—9.85186	+9.85180
(2)	+9.08999	9.07145	9.08662	9.07534	9.08318	+9.07889
Nos. (1)	—3.0598	+3.0686	—1.8755	+1.8778	—0.7110	+0.7109
—(2)	—0.1230	—0.1179	—0.1221	—0.1190	—0.1211	—0.1199
τ	—3.1828	+2.9507	—1.9976	+1.7588	—0.8321	+0.5910
(402) T	3.8172	9.9507	5.0024	8.7588	6.1679	7.5910
	3 49.03	9 57.04	5 0.14	8 45.53	6 10.07	7 3.546

(404) Middle $\tau = -0.120$ $T = 6.880 = 6^h\ 52^m.80$

Angles of Position.

(405) Constant	$+180°\ 0'$	$+180°\ 0'$
N	$+104\ 43$	$+104\ 20$
ψ	$+164\ 25$	$+\ 15\ 38$
	$+89\ 8$	$299\ 58$

Q First 89 to E. Second 60 to W.

(406, 407) Magnitude.

Δ Mean value, log 2.94427 Nos. 879.6

$$M = \frac{L - \Delta}{2s} = \frac{3270.0 - 879.6}{1787.8} = 1.338$$

The Moon in the Zenith.

	First Contact.	Last Contact.
(408) T	$5^h\ 0^m.14$	$8^h\ 45^m.53$
E	$+1\ .41$	$+1\ .44$
Constant	$12\ 0$	$12\ 0$
Sum	$17\ 1\ .55$	$20\ 46\ .97$
$-(\alpha - \alpha')$	$+3\ .72$	$-3\ .40$
Sum	$17\ 5\ .27$	$20\ 43\ .57$
ω (in arc)	$\begin{cases} 256°\ 18'\ \text{W.} \\ -103\ \ 42\ \text{E.} \end{cases}$	$\begin{cases} 310°\ 54'\ \text{W.} \\ -49\ \ 6\ \text{E.} \end{cases}$
(409) $\phi = \delta$	$12°\ 4'$. S.	$12°\ 33'$. S.

in the two columns headed *totality*, sin ψ will result greater than unity, and the work can therefore be carried no further. And in general throughout this whole theory of solar and lunar eclipses, when this results there is no eclipse.

There seems to be nothing difficult about this computation, especially to one familiar with solar eclipses. One-fiftieth of the $(\pi_1 + \pi' \pm s')$ is added on account of the earth's atmosphere, as above remarked. The angle ψ is required only for the shadow for use with the angles of position. The cosine is negative for beginning and positive for ending, which gives the quadrant for ψ and its sign results from the computation. The final times should not differ more than a few tenths of a minute from those of the first approximation, and usually the greatest difference is in the columns for penumbra, and the least differences for totality.

In the first approximation the times are distributed symmetrically about the middle time, as shown by differencing them; but in the second they are not, on account of the changes in all the semi-diameters, parallaxes, etc.

Professor CHAUVENET has not computed his example as accurately as it should have been, making but *one* approximation, and computing x and y at intervals of three hours. It seems also that the English, French, and German *Nautical Almanacs* are not so accurate as we are in this respect, since their times are symmetrical about the middle times, indicating but one approximation for the times.

The middle of the eclipse is already given in the computation for

totality, or in a partial eclipse for shadow. The mean of the two values for beginning and ending should be taken to give the time of the middle.

For the angles of position and magnitude the signs must be regarded as also in the foregoing work. For a partial eclipse, as above noted, the magnitude is less than unity. In the present example the moon is wholly obscured (unity), and its nearest limb 0.338ths of its diameter *within* the cone of shadow.

For the moon in the zenith, it is more convenient to reduce both $(\alpha - \alpha')$ and the equation of time to minutes and hundredths before interpolating. The former is a fraction of 1 hour and the latter a fraction of 24^h to the time T for beginning and ending of shadow. $(\alpha - \alpha')$ will always be $+$ for beginning and $-$ for ending, E is to be taken from page II. for the month in the *Nautical Almanac*, which reduces *mean* to *apparent* time. Finally, reduce the longitudes from time to arc.

192. *Lunar Appulse.*—This is the name given to a very close approach of the moon's limb to the earth's shadow without entering it. An occurrence of this kind took place in 1890, June 2. The quantity $(\pi_1 + \pi' - s')$ was about $2642''$. (I have not, however, the original figures to refer to.) One-fiftieth of this, $53''$, and the nearest approach of the limb of the moon to the shadow, was $19''.3$; but this latter quantity is not given in the *Nautical Almanac*. The value $53''$ is empirical and doubtful, so that I inserted the following clause: "The nearness of the approach and the uncertainty of the effect of the earth's atmosphere, render it doubtful whether the moon will enter the shadow of the earth or not."

A still more doubtful case occurred on December 11th of the same year, in which the computation resulted in an eclipse; but the duration of the partial eclipse was but $3^m\ 5^s.5$, and the magnitude 0.005, which is equivalent to but $9''.3$. Also in 1900, June 12, the magnitude was 0.001, equivalent to but $1''.9$.

The analytical condition for an appulse is that the magnitude, equation (407), is negative or

$$\Delta > L \qquad (410)$$

that is the distance of the moon's limb is greater than the radius of the shadow.

193. *Graphic Method.*—In the following manner a lunar eclipse may be projected, giving the times to one or two minutes. It is re-

marked first, that as we see the moon in the heavens, its motion is from the right hand toward the left, which is the positive direction in this method; and the positive direction for angles is from the north toward the left hand. We will take the eclipse previously computed, 1902, April 22.

A convenient scale for this projection is 1000″ to two-thirds of an inch, employed by the author for all the lunar eclipses he has computed. Fig. 27 is drawn to a scale of 1000″ to one inch, but reduced in the figure one half, or 2000″ to one inch.

First draw the axes of X and Y at right angles to one another, Fig. 27, Plate X., and lay off the negative values of x on the *right*, and the positive values on the *left* of the origin; on these points erect the ordinates: the positive above and the negative below (Art. 190). The shadow is a circle drawn from the origin of coördinates O with the radius (Art. 191).

$$L = \tfrac{61}{60}(\pi_1 + \pi' - s') = 2376''$$

and the penumbra a circle with the radius

$$L = \tfrac{61}{60}(\pi_1 + \pi' + s') = 4323''$$

which are mean values of those previously computed.

A line drawn through the ends of the ordinates is the path of the centre of the moon through the shadow, the ordinates marking the integral hours. With the dividers opened to the moon's semidiameter, $s = 894''$ move one leg along the path until the other leg is just tangent to the penumbral circle on the outside; these two points will be the first and last contacts of the moon with the penumbra. Likewise with the same radius, two circles drawn tangent to the shadow on the outside, will show the beginning and ending of the partial eclipse; and two other circles tangent to the shadow on the inside will show the beginning and ending of totality. If a line be drawn from the origin of coördinate perpendicular to the path, it will cross the path at a point I, which marks the middle of the eclipse. Describe a circle from this point with the same radius as the others.

The moon's path may now be divided up into 10-minute spaces, or even closer if the scale permits, and the centres of the small circles will mark the times of the phenomena. The point K, where the path crosses the axis of Y, is the time of opposition in right ascension.

The radii used above are the true radii of the shadow and penumbra, these quantities ± the moon's semidiameter s, used in the com-

putation, give the *centres* of the moon, which are necessary to find the times there, as well as in this graphic method. In case x and y have not already been computed, this projection may be made by the method for solar eclipses, Art. 161.

194. The various quantities employed in the general formulæ may also be shown in this projection. We will briefly recapitulate some of them; x and y have already been plotted as above. The hourly motions are the distance between abscissas and the differences of the ordinates, or very nearly so; and being *mean* hourly changes cannot be easily shown. But neglecting small terms, we have at the 9-hour ordinate erected from the point A, drawing $9\,B$ to the 10-hour ordinate, and parallel to the axis of X, $9B = x'$, and $B\,10 = y'$.

Also the 7-hour point being T_0, we have—

$$NOT_0 = M \qquad OT_0 = m$$
$$A\,9\,10 = NK9 = N \qquad 9 - 10 = n$$
$$OTC = (M - N)$$
$$OI = m \sin (M - N)$$

Connecting the centres of the moon $C\,C'$, with the centre of the shadow O in the formulæ.

$$OC = L \qquad OC' = L$$
$$OC4 = \psi = 164^\circ\ 25' \qquad OC'K = \psi = 15^\circ\ 38'$$
$$IC = IC' = L \cos \psi$$
$$IT_0 = m \cos (M - N)$$

and dividing these latter by n (the motion of the shadow in one hour), we have the times of describing these distances, and hence τ and T follow.

IC and IC' are here considered as equal, but the changing of all the quantities during the eclipse causes them to differ slightly, by an amount too small to be seen on a drawing. Also $x'\,y'\,n$ are here shown as the motions *between* two hours; in strictness, they are the motions *at* the given hour.

From the centres of the moon at first and last contacts erect the lines CP and $C'P'$ parallel to the axis of Y, then

$$\text{Angles of Position} \begin{cases} PCO = 89^\circ \\ P'C'O = 300^\circ \end{cases} \text{Measured toward the East or } \textit{left} \text{ hand.}$$

For the magnitude

$$OR = L, \text{ for totality.}$$
$$OI = m \sin (M - N) = \Delta, \text{ for middle of the eclipse.}$$
$$IR = L - \Delta$$

This latter, divided by the moon's diameter, gives the magnitude. It is seen that if $L = \Delta$, the path must be so far moved out that I falls upon R, the moon will be just tangent to the shadow on the *out*side, and $M = 0$. And if the moon is tangent to the *shadow* on the *in*side, the path at I is then distant from R by just $2s$, so that $M =$ unity, the limit between a partial and total eclipse.

SECTION XXV.

TRANSITS OF MERCURY AND VENUS ACROSS THE SUN'S DISK.

195. *Data, Elements.*—The formulæ for this phenomenon are derived from the fundamental formulæ for solar eclipses, with such modifications as the circumstances require, only the results being given in the present work. The data required from the *Nautical Almanac* are the right ascensions, declinations, parallaxes, and semi-diameters of both the sun and planet to be interpolated for three or four hours before and after the time of inferior conjunction. Also the sidereal time and equation of time from the *Almanac*, page II., for the month, and the distance of the planet from the earth at the time of inferior conjunction to compute or to verify the semidiameters and parallax. These latter, since the year 1900, are now given to two decimals for all planets, but formerly were given to only one decimal. The distance of the earth from the sun, log r', is required only to check the semidiameter, from which the constant of irradiation must be deducted.

The elements are given in the *Nautical Almanac*, similarly to those for eclipses (Art. 21).

The constant k is, as usual, the ratio of planet's semidiameter divided by that of the earth. The adopted semidiameters given in the *Nautical Almanac* for a number of years past are

$$\text{Mercury, } 3''.34 \qquad \text{Venus, } 8.546 \qquad (411)$$

at their mean distance from the sun, which is unity for each. The

earth's semidiameter at the mean distance, which is unity, is the sun's parallax π'.

The values of k given by Professor CHAUVENET; viz.:

$$\text{For Mercury, } 0.3897 \qquad \text{For Venus, } 0.9975 \tag{412}$$

depend upon ENCKE's parallax, which he uses throughout his chapter on Eclipses.

For the transit of Mercury, 1894, Nov. 10, the author of these pages used NEWCOMB's parallax, 8.848, which gave

$$\text{For Mercury, } 0.37748 \tag{413}$$

For the approaching transit of Mercury, 1907, Nov. 14, will undoubtedly be used the constant 8.800 of the Paris Conference, which gives

$$\text{For Mercury, } k = 0.37954, \log 9.57926 \tag{414}$$

This constant k is not used in the computations of transits as it is in eclipses, on account of the modifications made in the formulæ; it is, however, useful for verifying the semidiameter and parallax.

In equations of condition the constant k occurs, and various changes in the formulæ are necessary, but for this the reader is referred to CHAUVENET's own words, since this branch of the subject, both in eclipses and in transits, has not been considered in the present work.

196. *General Formulæ.*—The notation is the same as in eclipses, except where otherwise stated; a, δ, s, and other quantities, which in solar eclipses represent the moon, are here used to denote the planet, which, like the moon, comes between the earth and sun.

I. For the centre of the earth. (CHAUVENET, Arts. 356–58.)

$$\left.\begin{aligned} m \sin M &= (a - a') \cos \tfrac{1}{2} (\delta + \delta') \\ m \cos M &= (\delta - \delta') \end{aligned}\right\} \tag{415}$$

$\Delta a'$, Δa, $\Delta\delta'$, and $\Delta\delta$, being the hourly motions,

$$\left.\begin{aligned} n \sin N &= (\Delta a - \Delta a') \cos \tfrac{1}{2} (\delta + \delta') \\ n \cos N &= (\Delta\delta - \Delta\delta') \end{aligned}\right\} \tag{416}$$

$$\sin \psi = \frac{m \sin (M - N)}{s' \pm s} \tag{417}$$

In parts of an hour, $$\tau = \frac{s' \pm s}{n} \cos \psi - \frac{m}{n} \cos (M - N) \tag{418}$$

$$T = T_0 + \tau \tag{419}$$

$$Q = N + \psi \tag{420}$$

In the above formulæ $s' \pm s$ is the distance of the centres of the sun and planet, the upper sign for external contacts and the lower for internal. ψ is obtuse for ingress and acute for egress, as in eclipses. T_0 is the epoch hour and τ the correction for it. Q is the angle of position of the point of contact, the two values of ψ giving two values for Q.

If h represents 3600, the number of seconds in one hour, the above becomes

$$\text{In seconds,} \quad T = T_0 + h\left(\frac{s' \pm s}{n}\right)\cos\psi - \frac{hm}{n}\cos(M - N)^* \tag{421}$$

We also have, as in solar eclipses, for the distance and time of nearest approach of the centres,

$$T_1 = T_0 - \frac{hm}{n}\cos(M - N) \tag{422}$$

$$\Delta_1 = m \sin(M - N) \tag{423}$$

II. For any given place. By a method devised by LAGRANGE, which CHAUVENET gives as new, the times for any place are given by the following formula:

$$T' = T + \frac{\pi - \pi}{n \cos\psi}\left[\rho\cos\varphi'\sin D + \rho\cos\varphi'\cos D\cos(\theta - \omega)\right] \tag{424}$$

in which $T, n, \psi, D, \theta, \pi - \pi'$ are all constant, and given in the computation for the centre of the earth. T is the time computed for the centre of the earth, and $\rho \sin\varphi'$, $\rho\cos\varphi'$ the geocentric positions of the given place as used in eclipses (Art. 149).

For these constants compute as follows, taking the quantities from the computation for the centre of the earth for the proper times. For the *Nautical Almanac* these are computed but twice for ingress and egress at exterior contacts.

$$Q = N + \psi \tag{425}$$

$$\tan\gamma = \frac{\pi + \pi'}{\pi - \pi'}\tan\tfrac{1}{2}(s' \pm s) \tag{426}$$

$$\left.\begin{aligned} f\sin F &= \sin\gamma \\ f\cos F &= \cos\gamma\cos Q \end{aligned}\right\} \tag{427}$$

$$\left.\begin{aligned} \cos D\sin\left[A - \tfrac{1}{2}(a + a')\right] &= \cos\gamma\sin Q \\ \cos D\cos\left[A - \tfrac{1}{2}(a + a')\right] &= -f\sin\left[\tfrac{1}{2}(\delta + \delta') + F\right] \\ \sin D &= f\cos\left[\tfrac{1}{2}(\delta + \delta') + F\right] \end{aligned}\right\} \tag{428}$$

* CHAUVENET has a misprint here on page 598—the sign following T_0 should be $+$, not $-$ line 24.

Then the constants are

$$\theta = \mu - A \tag{429}$$

$$B = \frac{\pi - \pi'}{n \cos \psi} h \sin D \tag{430}$$

$$C = \frac{\pi - \pi'}{n \cos \psi} h \cos D \tag{431}$$

And equation (424) becomes

$$T' = T + B\rho \sin \varphi' + C\rho \cos \varphi' \cos (\theta - \omega) \tag{432}$$

For the sun in the zenith (Art. 189),

$$\left.\begin{aligned} \omega &= T + E - (a - a') \\ \varphi &= \delta' \end{aligned}\right\} \tag{433}$$

197. *Example.*—The example here given is from the author's computation sheets of the Transit of Mercury, 1894, November 10. The preliminary work, data, from the *Almanac* elements, etc., are omitted. This omission will not interfere with the understanding of the example or formulæ, since the data are so designated in the margin of the example.

As in eclipses, the work is kept compact, and quantities seldom rewritten which are previously given. In this respect he has acted upon a casual remark of one of his predecessors in several of the different portions of the *Nautical Almanac*—a gentleman known throughout the United States and Europe—" I never repeat a figure when it can be avoided."

In the first approximation, which is omitted from the example, the quantities are taken out of the tables of data for an hour near the middle of the transit. For this time, which is arbitrarily selected, it is convenient to select the exact instant of conjunction, T_0, for which time $M = 0$, $m = \delta - \delta'$, $a = a'$, and most of the other quantities are already gotten for the elements. The computation thus consists of but one column until ψ is reached, when the two values, $s' + s$ for external contacts, and $s' - s$ for internal, give two quite different values for sin ψ, and each of these will give two values for ψ, the obtuse for ingress, and the acute for egress, so that the first approximation now consists of four columns, as in the remainder of the work. The resulting times are given at the head of the four columns of the example; and for these times the quantities are taken from the tables of data for the final results. These times now become T_0 in each of the four columns, and are so noted in the margin. I have discarded CHAUVENET'S contractions, writing a_0 for $\frac{1}{2}(a + a')$, and restored the original quantities, making the formulæ clearer, though a little longer.

Transit of Mercury, 1894, Nov. 10.

I. For the Centre of the Earth G. M. T.

From 1st Approx.	T_0	$3^h\,55^m\,29^s.3$	$9^h\,12^m\,7^s.7$		$3^h\,57^m\,13^s.3$	$9^h\,10^m\,23^s.7$
Data	s'	969.80	969.85		969.80	969.85
Data	s	+4.94			—4.94	
	$s' \pm s$	974.74	974.79		964.86	964.91
Data	$\Delta\alpha$	—187.20	—186.62	$\Delta\delta$	+105.12	+105.17
Data	$\Delta\alpha'$	+151.80	+151.91	$\Delta\delta'$	—41.85	—41.68
	$(\Delta\alpha - \Delta\alpha')$	—339.00	—338.53	$(\Delta\delta - \Delta\delta')$	+146.97	+146.85
Data (415)	$\alpha - \alpha'$	+1009.81	—778.09		+1000.01	—768.31
Data	$\delta - \delta'$	—144.81	+630.54		—140.57	+626.30
Data	$\frac{1}{2}(\delta + \delta')$	—17° 18′ 6″	17° 15′ 19″		17° 18′ 5″	—17° 15′ 20″
	$\log(\alpha - \alpha')$	+3.00424	—2.89103		+3.00000	—2.88554
	$\cos \frac{1}{2}(\delta + \delta')$	+9.97989	+9.98000		+9.97989	+9.98000
(415)	$m \sin M$	+2.98413	—2.87103		+2.97989	—2.86554
	$m \cos M$	—2.16080	+2.79971		—2.14790	+2.79678
	$\tan M$	0.82333	0.07132		0.83199	0.06876
	M	+98 32 31	—49 41 0		+98 22 33	—49 31 0
	$\sin M$	9.99516	9.88223		9 99534	9.88115
	$\log m$	+2.98897	2.98880		2.98455	+2.98439
(416)	$\log(\Delta\alpha - \Delta\alpha')$	—2.53020	—2.52960			
	$n \sin N$	+2.51009	+2.50960			
	$n \cos N$	2.16723	2.16688			
	$\tan N$	0.34286	0.34272			
	N	—65 34 40	—65 34 14		—65 34 40	—65 34 14
	$\sin N$	9.95929	9.95926			
	$\log(1 : n)$	+7.44920	+7.44966		+7.44920	+7.44967
(417)	$M - N$	+164 7 11	+15 53 14		+163 57 13	+16 3 14
	$\sin(M - N)$	+9.43716	+9.43734		+9.44156	+9.44176
	$\cos(M - N)$	—9.98310	+9.98308		—9.98274	+9.98272
Least distance	$m \sin(M - N)$	+2.42613	+2.42614		+2.42611	+2.42615
	$s' + s$	+2.98889	2.98891	$s' - s$	2.98447	+2.98449
	$\sin \psi$	+9.43724	9.43723		9.44164	+9.44166
	ψ	+164 7 0	+15 52 59		+163 57 3	+16 3 0
(418)	$\cos \psi$	—9.98309	+9.98309		—9.98273	+9.98273
	log (1) × 3600	—3.97748	+3.97796		—3.97270	+3.97319
	log (2) × 3600	—3.97757	+3.97784		—3.97279	+3.97307
	—Nos. (2)	+2 38 16.6	—2 38 22.5		+2 36 32.8	—2 36 38.7
(422) Middle	$=T_1$	6 33 45.9	6 33 45.2		6 33 46.1	6 33 45.0
	Nos. (1)	—2 38 14.7	+2 38 25.2		—2 36 30.7	+2 36 41.4
Final times.	T	3 55 31.2	9 12 10.4		3 57 15.4	9 10 26.4
Contacts.		I. Ingress.	IV. Egress.		II. Ingress.	III. Egress.

(422) Middle for *Nautical Almanac* from 1st Approx. $6^h\,33^m\,48^s.5$

(423) Least distance of centres, $m \sin(M - N)$ 4 26.76

TRANSIT OF MERCURY, 1894, NOV. 10.

Constants.

	T	$3^h\ 55^m\ 31^s.2$	$9^h\ 12^m\ 10^s.4$
Data at ☌	$\pi + \pi'$	$= 13''.08 + 8'.94$	$22''.0$
	$\pi - \pi'$	$=$	$4\ .14$
	$s' + s$	$= 4\ .94 + 16\ \ 9''.83 = 16\ 14\ .77$	
	$\frac{1}{2}(s' + s) =$		$8\ \ 7\ .38$

(426) $\log \pi + \pi'$	1.34282	
$\tan \frac{1}{2}(s' + s)$	7.37343	
	8.71625	
$\pi - \pi'$	0.61700	
$\tan \gamma$	8.09925	
$\sin \gamma$	+8.09922	
$\cos \gamma$	+9.99997	
(425) $Q = N + \psi$	+98 32 20	−49 41 15
$\sin Q$	+9.99516	−9.88226
$\cos Q$	−9.17167	+9.81087

(427) $f \sin F$	+8.09922	+8.09922
$f \cos F$	−9.17164	9.81084
$\tan F$	8.92758	8.28838
F	+175 9 43	+1 6 46
cos	9.99845	9.99992
$\log f$	+9.17319	9.81092
(428) $\frac{1}{2}(\delta + \delta') + F$	+157 51 37	−16 8 33
sin	+9.57619	−9.44409
cos	−9.96674	+9.98253
$\cos D \sin (\ )$	+9.99513	−9.88223
$\cos D \cos (\ )$	−8.74938	9.25501
$\tan (\ )$	1.24575	0.62722
$A - \frac{1}{2}(a + a')$	+93 15 1	−76 43 30
sin	9.99930	9.98824
$\cos D$	+9.99583	+9.89399
$\sin D$	−9.13993	+9.79345

	T	$3^h\ 55^m\ 31^s.2$	$9^h\ 12^m\ 10^s.4$
	Contacts.	I. Ingress.	IV. Egress.
	(430) $(\pi - \pi') \times 3600$	+4.17330	+4.17330
	$n \cos \psi$	−2.53389	+2.53343
		1.63941	1.63987
	$\log B$	+0.77934	+1.43332
	(431) $\log C$	−1.63524	+1.53386
See Text.	μ_1 (noon)	$15^h\ 18^m\ 31^s.4$	$15^h\ 18^m\ 31^s.4$
	T	3 55 31 .2	9 12 10 .4
	sid. interval	38 .7	1 30 .7
	μ (time)	19 14 41 .3	0 32 12 .5
	(429) μ (arc)	288° 40′ 19″.5	8° 3′ 7″
	$\frac{1}{2}(a + a')$	225 57 3	225 55 30
	$A - \frac{1}{2}(a + a')$	+93 15 1	−76 43 30
	A	319 12 4	149 12 0
	θ	329 28 15	218 51 7

Sun in the Zenith.

(433) T	$3^h\ 55^m.52$	$9^h\ 12^m.17$
E	+15 .92	+15 .90
	4 11 .44	9 28 .07
$-(a - a')$	−1 .13	+0 .86
	4 10 .31	9 28 .93
ω in arc	62° 35′ W.	+142° 14′ W.
(433) ϕ	−17° 17′ S.	−17° 21′ S.

Angles of Position.

Q	98° 32′ to E.	−49° 41′ to W.

Final Equations for the Times for any Place.

$$(432)\left\{\begin{aligned} T' &= 3^h\ 55^m\ 31^s.2 + [0.77934]\,\rho \sin \phi' - [1.63524]\,\rho \cos \phi' \\ &\qquad \cos (329^\circ\ 28'\ 15'' - \omega) \\ T'' &= 9\ \ 12\ \ 10\ .4 + [1.43332]\,\rho \sin \phi' + [1.53386]\,\rho \cos \phi' \\ &\qquad \cos (218^\circ\ 51'\ 7'' - \omega) \end{aligned}\right.$$

Owing to the *retrograde* motion of Mercury at conjunction inferior, $\alpha - \alpha'$ has a contrary sign to that in eclipses, being + for beginning and — for ending; and the planet is seen passing over the sun's disk from left to right. The changes in the hourly motions are so small that the quantities for the I. and II. contacts are the same; likewise those for the III. and IV. In the computations the group beginning with $\Delta\,\alpha$ belongs as well to the two last columns as to the first. $\Delta\,\delta$ also belongs to the two first columns, they are so placed merely to economize space. N, which follows, is computed once for beginning and once for ending.

The terms (1) and (2) are multiplied by 3600, by its logarithms, while adding the other logarithms, giving the quantities in seconds, which are then reduced to hours and minutes. Each column gives a value for the middle—the second term; but the more accurate value is that derived from the first approximation, which is made for a time not far from the true middle.

A partial check upon the computation may be had from the distances between the centres of the sun and planet, so that we have at the instant of contact—

$$m = s' + s \quad \text{or} \quad m = s' - s \tag{434}$$

Thus in the present example for the four contacts:

	I.	IV.	II.	III.
m	974.92	974.54	965.05	964.67
$s' \pm s$	974.74	974.79	964.86	964.91
Discrepancy	+18	−0.25	+0.19	−0.24

The discrepancies here noticed may arise partly from the formulæ for m being only approximate (Arts. 127 and 173), but more likely from the error of the two right ascensions when reduced from time to arc, which may amount to 0″15 in each (Arts. 19 and 29). But we may notice that the contacts I. and II. are so near together in time that they would usually be affected by the *same error* and with the *same sign*, and similarly for III. and IV.

The computation for the constants presents nothing especially difficult; $\pi\,\pi'$ $s\,s'$ are taken from the elements for the time of conjunction. They are almost constant during the continuance of the transit. Quantities required here, which are given in the former part of the work, are generally not repeated, for example, N and ψ are given above; but Q only is set down here. The work thus stands exactly as on my computing sheets, except the computation

for the sidereal time, which I have enlarged a little, by inserting the mean time T, the quantity $\frac{1}{2}(a + a')$ from the data and that which follows it from the previous work. The sidereal time μ is the sidereal time at noon plus the mean time of the phenomenon reduced to sidereal time. a and a' must be interpolated separately from the data, and the mean $\frac{1}{2}(a + a')$ gotten; then with the time $[A - \frac{1}{2}(a + a')]$ from the first computation, A is found.

The sun in the zenith is found as in Lunar Eclipses, Art. 189. The points of the earth's surface thus found give the hemisphere approximately from which the contact may be seen, and in a stereographic projection a circle described from this point with the radius of the sphere will show the hemisphere. A little more than a hemisphere may then be projected stereographically.

198. *Graphic Representation.*—This is not of as much value as in eclipses, except as a matter of instruction. The abscissas and ordinates may be laid off from formula (415), thus:

$$\left.\begin{aligned} x &= m \sin M = (a - a') \cos \tfrac{1}{2}(\delta + \delta') \\ y &= m \cos M = (\delta - \delta') \end{aligned}\right\} \qquad (435)$$

by which the path of the planet may be drawn, remembering to lay off the positive values of x on the *left*, and the negative values on the *right*, of the origin for direct vision, since the motion of the planet is *retrograde* at inferior conjunction. The positive direction for angles is from the north point toward the *left*. The radii of the sun and planet are their semidiameters. Mercury will be a mere dot on the large surface of the sun.

From this graphic projection the various quantities used in the formulæ may be shown, as explained in Section V., the Extreme Times Generally in Solar Eclipses.

The only charts of a transit that have been given in the *Almanac* are those for Venus in 1882. They differ considerably from charts of an eclipse, the shadow moving from east to west. Limiting curves would rarely occur, and the rising and setting curves are nearly great circles of the earth intersecting more or less at right angles; so that the whole transit is visible to about one-fourth of the earth, and invisible to the opposite fourth.

199. Transits of Mercury and Venus are especially employed in determining the solar parallax, for which Venus gives much more accurate results than Mercury, on account of its greater proximity

to the earth. Professor CHAUVENET has developed this method in his chapter, already quoted from, to which the reader is referred.

In the "*Papers relating to the Transit of Venus* in 1874, prepared under the direction of the Commission authorized by Congress," Part II., are given the formulæ devised by Dr. GEORGE W. HILL, previously referred to with charts of the transit.

And in the "Instructions for Observing the Transit of Venus, December 6, 1882," are similar charts prepared by Dr. HILL. The curves are similar to the outline curves in eclipses, but drawn at intervals of one minute. There is also given an orthographic projection of the earth by the author of these pages, showing the path of the several places of observation over the sphere during the continuance of the transit, the paths and circles of the earth projecting into true ellipses.

In Part I. of the "Observations on the Transit of Venus, December 8–9, 1874," by Professor NEWCOMB, 1880, the observations of the various parties of observers are discussed. There is also given by the author of these pages a description and detailed drawings of the photoheliograph used for reflecting the sun's image upon the photographic plate. This instrument is permanently mounted at the Naval Observatory.

In the *Astronomical Papers of the American Ephemeris and Nautical Almanac*, i., Part VI., "Discussion and Results of Observations on the Transits of Mercury, 1677 to 1881," Professor NEWCOMB has, among other things, devoted one section on the law of the recurrence of these transits, giving diagrams and tables of the transits covering several centuries.

SECTION XXVI.

OCCULTATIONS OF FIXED STARS BY THE MOON.

200. *General Formulæ* (*Longitude*).—The occultation of a fixed star by the moon is but a special case of a solar eclipse, in which the sun is supposed to be at an infinite distance, and represented by the star, its parallax and semidiameter are zero and its distance r' infinity. The cone of shadow then becomes a cylinder, whose radius is k. This is shown by equations (36) and (38), but equation (35) takes the indeterminate form. It is better seen in the equations as given in Art. 182: sin f is zero, its cosine unity, and there is left $l = k$.

And as shown in Section XXI., a larger value of this constant is required in occultations than in eclipses, to allow for the increased semidiameter of the moon caused by the lunar mountains, irradiation, etc. This was apparently first discovered in the discussions of equations of condition from observations of occultations.

The general equations which CHAUVENET has given in Art. 341 are under the form for correcting the longitude of a place from the observed times. This is the chief purpose for which the formulæ in this section are generally used. They are as follows, the notation being the same as for solar eclipses, except where otherwise specified.

$$\left.\begin{aligned} x &= \frac{\cos\delta\sin(\alpha-\alpha')}{\sin\pi} \\ y &= \frac{\sin(\delta-\delta')\cos^2\tfrac{1}{2}(\alpha-\alpha')+\sin(\delta+\delta')\sin^2\tfrac{1}{2}(\alpha-\alpha')}{\sin\pi} \end{aligned}\right\} \quad (436)$$

Hour angle $\qquad \vartheta = \mu - \alpha' \qquad (437)$

$$\left.\begin{aligned} A\sin B &= \rho\sin\varphi' \\ A\cos B &= \rho\cos\varphi'\cos\vartheta \end{aligned}\right\} \quad (438)$$

$$\left.\begin{aligned} \xi &= \rho\cos\varphi'\sin\vartheta \\ \eta &= A\sin(B-\delta') \\ \zeta &= A\cos(B-\delta') \end{aligned}\right\} \quad (439)$$

If $\log\zeta$ is small, correct the last three for refraction by the Table (Section XXII.).

$$\left.\begin{aligned} m\sin M &= x_0-\xi \\ m\cos M &= y_0-\eta \end{aligned}\right\} \quad (440)$$

$$\left.\begin{aligned} n\sin N &= x' \\ n\cos N &= y' \end{aligned}\right\} \quad (441)$$

For k, PETER's value (Art. 182), $k = 0.272518$
American Ephemeris, $k = 0.272506$ $\qquad (442)$

$$\sin\psi = \frac{m\sin(M-N)}{k} \quad (443)$$

If the local mean time t was observed, take $h = 3600.00$.

$$\tau = \frac{hk\cos\psi}{n} - \frac{hm\cos(M-N)}{n} \quad (444)$$

$$\omega = T_0 - t + \tau \quad (445)$$

If the local sidereal time μ was observed, take $h = 3609.856$.

$$\omega = \mu_0 - \mu + \tau \quad (446)$$

μ_0 being the sidereal time at the first meridian.

The value of k above given is the more recent determination of

PETERS, and is a little smaller than OUDEMAN'S, given by CHAUVENET.

To determine the longitude of a place, the occultation must also be observed at some other place whose longitude is known, and all the observations must then be combined to form the equations of condition; for which the reader is referred to CHAUVENET'S own article, as it is the scope of the present work to include only such computations as are necessary for compiling the *Nautical Almanac* and for general prediction. For this reason the above formulæ would have been omitted, were it not for the fact that some of them are referred to in previous pages, as well as in the succeeding pages of this section; and, moreover, in order that the connection of these few may be seen, the whole series is given.

201. *Prediction for a Given Place.*—This is of use in order to be prepared to observe an occultation, especially if by the dark limb of the moon, for either immersion or emersion. The time of conjunction may be found by the method used for solar eclipses (Art. 21), from the differences of the quantity $(\alpha - \alpha')$, in which α' for the star is a constant in formula (11).

$$T_0 = \frac{-(\alpha - \alpha')}{\Delta_1 - \frac{1}{2}\Delta_2 - \frac{1}{2}\Delta_2\left(\frac{\alpha - \alpha'}{\Delta_1}\right)} \tag{447}$$

CHAUVENET'S formulæ, Art. 345, are as follows:

The epoch hour T_1, being optional, is selected for the first approximation at the instant of conjunction in right ascension, found above. For this time $(\alpha - \alpha') = 0$, whence equation (336) becomes

$$\left.\begin{aligned} x &= 0 \\ y &= \frac{\sin(\delta - \delta')}{\sin \pi} = \frac{\delta - \delta'}{\pi} \text{ nearly.} \end{aligned}\right\} \tag{448}$$

We also have from the "Diff. for 1 Minute," in the *Nautical Almanac*, $\Delta\alpha$ and $\Delta\delta$, the hourly motions are

$$\left.\begin{aligned} x' &= 900\ \frac{\Delta\alpha}{\pi} \cos\delta \text{ (given in arc)} \\ y' &= \ \ 60\ \frac{\Delta\delta}{\pi} \end{aligned}\right\} \tag{449}$$

At the time of contact the hour angle

$$\vartheta = \mu_1 - \alpha' - \omega \tag{450}$$

in which μ_1 is the sidereal time at the first meridian, α' the right ascension of the star, and ω the longitude of the given place. We also have

$$\left.\begin{aligned} A \sin B &= \rho \sin \varphi' \\ A \cos B &= \rho \cos \varphi' \cos \vartheta \end{aligned}\right\} \tag{451}$$

$$\left.\begin{aligned} \xi &= \rho \sin \varphi' \sin \vartheta \\ \eta &= A \sin (B - \delta') \end{aligned}\right\} \tag{452}$$

$$\mu' = 54148 \sin 1'' \qquad \log \mu' = 9.41916 \tag{453}$$

$$\left.\begin{aligned} \xi' &= \mu' A \cos B \\ \eta' &= \mu' \xi \sin \delta' \end{aligned}\right\} \tag{454}$$

$$\left.\begin{aligned} m \sin M &= x - \xi \\ m \cos M &= y - \eta \end{aligned}\right\} \qquad \left.\begin{aligned} n \sin N &= x' - \xi' \\ n \cos N &= y' - \eta' \end{aligned}\right\} \tag{455}$$

As now adopted by the *Nautical Almanac* office for occultations,

$$k = 0.272506 \qquad \log k = 9.43538 \tag{456}$$

$$\sin \psi = \frac{m \sin (M - N)}{k} \tag{457}$$

$$\tau = \frac{k \cos \psi}{n} - \frac{m \cos (M - N)}{n} \tag{458}$$

$$T = T_1 + \tau \tag{459}$$

The angle ψ is obtuse with its cosine negative for immersion, and acute with its cosine positive for emersion, all the quantities in the first approximation being taken for the time T_1.

Angles of Position.—In solar eclipses these were measured on the sun's limb toward the moon, but here we wish the angle measured on the moon's disk toward the star; it will therefore differ 180° from the eclipse value. Hence, we have the angles of position from the north point.

$$Q = 180° + N + \psi \tag{460}$$

And from the vertex

$$\left.\begin{aligned} p \sin \gamma &= \xi + \tau\xi' \\ p \cos \gamma &= \eta + \tau\eta' \end{aligned}\right\} \tag{461}$$

$$\text{Angles of position from the vertex} = 180 + N + \psi - \gamma \tag{462}$$

The distance of the star from the moon's limb l at the middle of the occultation is found thus. The distance of the star from the centre of the moon is as in former cases.

$$\Delta = \pm m \sin (M - N) \tag{463}$$

in which Δ is to be taken as positive. Δ is a fraction of the unit of measure throughout this section, which is π (see Section XXI.);

hence in seconds of arc it is $\pi\Delta$. The moon's semidiameter being s, the distance from the moon's limb is $s - \pi\Delta$; but $s = k\pi$, therefore the distance from the moon's limb, omitting the augmentation, is

$$\pi(k - \Delta)^* \tag{464}$$

For a second approximation the times resulting from the first approximation become T_1, for which the quantities must be again taken from the tables. x and y must then be computed for these times from equation (436), or they may be computed by the following:

$$\left.\begin{aligned} x &= x_0 + x'(T_1 - T_0)\dagger \\ y &= y_0 + y'(T_1 - T_0) \end{aligned}\right\} \tag{465}$$

And ϑ may be found from τ, by reducing it from mean time to sidereal, and then from time to arc, and adding this to the value of ϑ, first found. CHAUVENET uses four-place logarithms here, but five-place would be better, though the labor would also be greater.

If in any computation it is found that $\sin\psi > 1$, it shows that there is no occultation, for then we have

$$\Delta > k \tag{466}$$

No example of the above formulæ is here given, but the reader will find an example given at the end of each *Nautical Almanac*, in the addenda, "On the Use of the Tables," the computation being carried out in full with various precepts and explanations.

202. *The Limiting Parallels.*—These are intended to show approximately the portion of the earth's surface over which an occultation will be visible, by finding the extreme parallels of latitude between which the path across the earth lies. The intersection of the cylinder of rays from the moon, with the fundamental plane, we will call, for want of a better word, the *shadow*. As in eclipses, the shadow moves across the earth from west to east, forming the shadow path. In Fig. 28, Plate XI., the large circle represents the earth, orthographically projected upon the fundamental plane, the shadow path being that portion included between the lines $a\,a'$ and $b\,b'$.

In occultations, the unit of measure for the earth's radius is π, the moon's parallax (Section XXI.); the moon or shadow is k, a linear fraction of π, which, reduced to seconds, becomes $k\pi$, the moon's semidiameter. It is seen in Fig. 28 that if the shadow falls so far north

* CHAUVENET has transposed k and Δ in his text, so that his equation will give *negative* values, because k, the radius of the shadow, must be the greater of the two numerically.

† Through some oversight in CHAUVENET's text (Art. 345, p. 557) the term x_0 is omitted in these formulæ.

of the earth that the southern line $b\,b'$ is tangent to the earth on its northern part, the distance of the centres from one another will be

$$\left.\begin{aligned}\pi + s &= 61'\,28''.8 + 16'\,44''.4 = 1^\circ\,18'\,13''.2\\ &= \text{nearly } 1^{\circ\prime}\,20'\end{aligned}\right\} \qquad (467)$$

which expresses the approximately extreme difference in declination between the moon and a star, within which an occultation is visible. This limit is taken in round numbers at $1^\circ\ 20'$.

Not every star in the heavens can be occulted. The inclination of the moon's orbit to the ecliptic is about $5^\circ\ 9'$, and when at its greatest latitude north, a star $1^\circ\ 20'$ *south* of it may cast a shadow at the limit on the north pole, while a star $1^\circ\ 20'$ *north* of it will cast a shadow at the limit on the south pole, so that the extreme distance of a star from the ecliptic, in order to be occulted, is

$$5^\circ\ 9' + 1^\circ\ 20' = 6^\circ\ 29'.$$

Only those stars lying within this distance of the ecliptic can be occulted. None others. As the node of the moon's orbit has a retrograde motion on the ecliptic, the extreme north and south points of the orbit move slowly through this celestial belt, making a complete revolution during the Saros of 18 years, or 242 lunations.

Formerly, occultations for the *Nautical Almanac* were computed for even faint stars, but for 1905 the limit of magnitude was fixed at 6.5, and a catalogue of stars made including all that can possibly be occulted down to this magnitude. I do not know, however, the actual limits taken for this star catalogue.

In computing the limiting parallels, the first thing done is to ascertain what stars will be occulted during the year. With the advanced pages of the moon's ephemeris, as given in the *Almanac*, these are compared with the above-mentioned catalogue of stars, keeping the right ascensions equal, and noting any star whose declination differs from that of the moon by an amount not greater than $1^\circ\ 20'$. With a little practice this can be done quite rapidly, including all doubtful stars for closer investigation afterward, and making a list of their names.

The limiting parallels are then computed by the following formulæ, given by CHAUVENET, Art. 346.

First find the time of conjunction by equation (447), for which the moon's right ascension for four hours must be taken from the *Nautical Almanac* pages, for which time the quantities in the formulæ must be interpolated.

$$y_0 = \frac{\delta - \delta'}{\pi} \qquad (468)$$

$$\left.\begin{aligned} x' &= 900 \frac{\Delta\alpha}{\pi} \cos\delta \\ y' &= \ \ 60 \frac{\Delta\delta}{\pi} \end{aligned}\right\} \tag{469}$$

$$\tan N = \frac{x'}{y'} \qquad N < 90^\circ \tag{470}$$

And the limiting parallels φ_1 and φ_2 are

$$\left.\begin{aligned} \cos\gamma_1 &= y_0 \sin N \pm 0.2723 && (\gamma_1 < 180^\circ) \\ \sin\beta &= \sin N \cos\delta' && (\beta < 90^\circ) \\ \varphi_1 &= \beta \pm \gamma_1 \end{aligned}\right\} \tag{471}$$

$$\left.\begin{aligned} \cos\gamma_2 &= y_0 \sin N \mp 0.2723 \\ \sin\varphi_2 &= \sin(N \mp \gamma_2) \cos\delta' && N < 90^\circ \end{aligned}\right\} \tag{472}$$

If the star's declination is *north*, the upper signs are to be used throughout, and φ_1 is the northern and φ_2 the southern parallel. If the declination is *south*, use the lower signs, and φ_1 is the southern and φ_2 the northern parallel.

N is to be considered as positive and less than 90°, although the angle may strictly be obtuse, the reason for which will be seen below.

To the above are also appended the following precepts: When all the shadow *does not fall upon the earth*, as shown by $\cos\gamma_1$ or $\cos\gamma_2$ being imaginary.

(1) *Cos* γ_1 *Imaginary.*—The occultation is visible beyond the *elevated* pole (north or south), and $+90$ in north latitude or -90 in south latitude is the conventional designation that the occultation is visible *beyond* the pole for a distance in degrees equal numerically to the star's declination.

(2) *Cos* γ_2 *Imaginary.*—The occultation is visible in the vicinity of the *depressed* pole (north or south). If the star's declination is north, the south pole is depressed and the limiting parallel is $\varphi_2 = \delta' - 90^\circ$. If the declination is south, the north pole is depressed and the limiting parallel is $\varphi_2 = \delta' + 90^\circ$. The algebraic signs of δ' must here be regarded.

(3) If $\varphi_1 = \beta \pm \gamma_1$ exceeds 90°, the true value is either $\varphi_1 = 180 - (\beta \pm \gamma_1)$ or $\varphi_1 = -180^\circ - (\beta \pm \gamma)$, since these have the same sine.

203. *Example.*—We will take the star 46 Virginis and the date 1905, Sept. 1. This example includes a little more than the limiting parallels, for it was computed for the *Nautical Almanac*, and gives all the elements of the occultation as usually published.

First take from the moon's ephemeris the right ascension for four hours, two before and two after the approximate time of conjunction;

subtracting from these the star's reduced right ascension gives $\alpha - \alpha'$, with which the correct Greenwich mean time of conjunction is to be gotten by formula (447), which time is placed at the head of the example. The Washington mean time is found by deducting from it the longitude of Washington in time. This is then reduced to sidereal time (s. t.). The hour angle ϑ is then found by (473); the sidereal time at Greenwich noon is corrected for the sidereal interval corresponding to the longitude $5^h\ 8^m.26$ from Table III., at the end of the *Nautical Almanac*, giving the sidereal time at Washington mean noon. Adding to this the above $8^h\ 26^m.2$ gives the Washington mean time of conjunction.

OCCULTATION OF 46 VIRGINIS, 1905, SEPT. 1.

LIMITING PARALLELS AND DATA FOR THE *Nautical Almanac*.

[References to pages are to the *Nautical Almanac*, 1905.]

Elements for *Nautical Almanac*.

Star's name, Magnitude, 6.1, 46 Virginis.

(447) ☌ G. M. T.	$13^h\ 33^m.1$
ω Washington, p. 524	5 8 .26
N. A. ☌ W. M. T. (N. A., p. 470)	8 24 .8
Reduc. to s. t., p. 592	+1 .4
☌ in s. t.	8 26 .2
s. t. at G. M. N., p. 147	$10^h\ 39^m.92$
Reduc. for 5^h 8.26, p. 592	0 .85
s. t. W. M. N.	10 40 .8
W. M. T. of ☌	19 7 .0
N. A. ✳ R. A. (mean place)	$12^h\ 55^m\ 42^s.357$
N. A. $\Delta\alpha$	+0 .560
✳ Reduced R. A., α'	12 55 42 .92
N. A. ✳ Dec. (mean place)	$-2°\ 51'\ 27''.78$
N. A. $\Delta\delta$	−0 .94
N. A. ✳ Reduced Dec. δ'	−2 51 28 .72
(473) ✳ R. A. as above.	α $12^h\ 55^m.7$
N. A. Hour angle. $H =$ ☌ $- \alpha$	+6 11 .3
☽ Dec. at 13^h	$-1°\ 54'\ 47''.7$
Interpol. for ☌	−6 38 .3
δ	−2 1 26 .0
$\delta - \delta'$	+0 50 2 .7

For Limiting Parallels.

(468) log $(\delta - \delta')$	3002″.8	+3.47752
log π		3.56003
N. A. y_0	+0.8270	+9.91749
(469) Diff. for 1^m	$2^s.3390$	0.36903
$\cos\delta$		9.99973
$15 \times 60 = 900$		2.95424
In arc.		3.32300
π		3.56003
N. A. x'	0.5794	9.76297
Diff. for 1^m	−12″.027	−1.08016
60		1.77815
$1 : \pi$		6.43997
N. A. y'	−0.1987	−9.29828
(470) $\tan N$		0.46469
N (acute)		+71° 4′
(471) $\sin N$		+9.97584
$y_0 \sin N$	+0.7822	+9.89333
k	−0.2723	
$\cos\gamma_1$	+0.5099	
γ_1	+59° 20′	
$\sin N$		9.97584
$\sin N \cos\delta' = \sin\beta$		9.97557
β	+70 58	
$\phi_1 = \beta - \gamma_1$	+11 38	
(472) $\cos\gamma_2$	1.0545	Imaginary.
Case 2.		

$\phi_2 = \delta' + 90° = -2°\ 51' + 90°$
$= +87°\ 9'$

The star's *mean* right ascension and declination (which are placed in the *Almanac* on page 445) are then reduced to the given date for proper motion, annual variation, precession, etc. With the star's right ascension first find the hour angle as gotten for the *Almanac*.

$$\vartheta \text{ or } H = ♂\ W.M.N - a' \qquad (473)$$

The moon's declination is interpolated for the time of conjunction and $\delta - \delta'$ then gotten. This completes the first column of the example, which gives chiefly the data to be inserted in the *Almanac* (pages 445 and 470), and also the data to be used in the formulæ for the limiting parallels.

Formulæ (468–72) are now to be computed for the limiting parallels. It is first to be noted that Δa and $\Delta\delta$ in the first column of the example are the *Nautical Almanac* notation for the small reductions of the star's place, and are given on page 470 of the *Almanac* under that notation. But Δa and $\Delta\delta$ in the second column of the example and in formulæ (467) are quite different quantities, being the "diff. for 1 minute" of the moon's motion given in the *Nautical Almanac* on page 150. N, the angle which the path makes with the axis of Y, is to be taken less than 90°, though the formulæ may strictly give it obtuse, as they do in this example. The constants 900 and 60 are 15, to reduce the right ascensions from time to arc, and 60 to reduce Δa and $\Delta\delta$ from the motion in one minute to the motion in one hour, so that x' and y' can be given in the *Almanac* as *hourly* motions. Five-place logarithms are used here, and y_0, x', y' are given to four places of natural numbers for the *Almanac*.

In formulæ (471) and (472), the star's declination being south, the *lower* signs are to be used, and φ_1 is the *southern* parallel $+ 11°$. γ_1 may be obtuse, but in the example it is acute.

γ_2 results $>$ unity, which comes under Case 2. The declination being south, the north pole is depressed, the shadow extends beyond the earth, and the parallel furthest from the equator, which is above (or in) the fundamental plane, is $\delta + 90° = + 87°$.

For this example, besides various other suggestions, I am indebted to Mr. JAMES ROBERTSON, of the *Nautical Almanac* office, who now computes the *Occultations*, and who kindly placed his computing sheets at my disposal. Mr. ROBERTSON has transformed CHAUVENET'S formulæ and devised a shorter method for these limiting parallels, with the use of manuscript tables; and on that account I have been obliged to deviate from the latter part of his work and keep to CHAUVENET'S formulæ. It is hoped that he will publish his method and tables.

204. *Graphic Representation.*—The declination in the above example is so small, all parts of the construction following could not well be seen, so that for illustration I have selected another occultation from the *Nautical Almanac*, 1905, page 450, Feb. 10.

85 *Ceti* Mag. 6.3	☌ $21^h\ 29^m.3$	$\delta' = +10°\ 20'.1$
$y_0 = +0.2043$	$x' = +0.5297$	$y' = +0.1461$
Limiting parallels,	$+47°$ and $-19°$	

In Fig. 28, Plate XI., let the larger circle represent the orthographic projection of the earth upon the fundamental plane. The radius of the circle is the unit of measure, and in arc is the moon's parallax π. Z is the zenith of the projection, as in previous figures. The shadow passes from left to right over the earth, and at the time of conjunction is at the point K on the axis of Y. ZK therefore represents $\delta - \delta'$ in arc proportionate to the radius of the earth's sphere, or dividing by π, it is a decimal fraction, the quantity $y_0 = +0.2043$ in formula (468). If we lay off $ZA = x' = +0.5299$, the motion of the shadow in right ascension in one hour, and erect $AD = y' = +0.1461$ (equation (469)), the motion in declination in one hour, ZD will represent the direction of the shadow path. Drawing a line through K parallel to this last, we have MR, the path of the shadow across the earth. If two other lines aa' and bb' be drawn parallel to MR and distant from it $= k = 0.2723$, they will be the limiting lines of the shadow path. Let P be the north pole, elevated in this figure, which shows that the star's declination is *north*, then $PKR =$ the angle N (equation (470)); also, $PZQ = N$.

If the plane of YZ, which is perpendicular to the plane of the paper, be revolved about the axis of Y through an angle of 90° to the right, Z will fall at F, NS will represent arc $Na\ bs$ in the fundamental plane, and the pole P will fall at some point P'. We can therefore lay off any desired declination. $NZP' = \delta' = +10°\ 20'$, and when revolved back, P' will fall at P, whose position we now know. It is seen that as ZN or ZP' is taken as unity, ZP will be the cosine of the declination.

Drawing ZH parallel to the path and also ZI and PQ perpendicular to the path, we have

$$ZI = ZK \sin N = y_0 \sin N$$

Formulæ (471) and (472) should be closely followed in connection with these pages. We have also the following quantity, which is arbitrarily called $\cos \gamma_1$.

$$ZC = \cos \gamma_1 = y_0 \sin N + 0.2723 \qquad \text{(Equa. (471))}$$

We use the + sign because the declination is north.

Also, $$PQ = PZ \sin N$$

or $$\sin \beta = \sin N \cos \delta' \qquad \text{(Equa. (471))}$$

The angles γ_1 and β will now be shown and their sum. Draw Za, which is the radius of the sphere; CZ, being the cosine of γ_1, CZa must be the angle γ_1 itself. Also, since PQ or its equivalent $OZ = \sin \beta$, draw OG, parallel to the path, passing through P of course, and meeting the circle in G, then $GZH = \beta$. This angle must be revolved round the point Z so that G will fall in C'; therefore, laying off the arc $HT = GC'$, we have $\beta = C'ZT$; and γ_1, now laying adjacent thereto, $aZT = \beta + \gamma_1$, an obtuse angle in which by Case III. φ_1 is its supplement, which exactly equals 47°, given in the *Nautical Almanac;* its sine is the perpendicular dropped from T upon aZ produced.

The southern parallel is found thus:

$$\cos \gamma_2 = y_0 \sin N - 0.2723 = ZE,$$

a negative quantity in this example which gives the obtuse angle $C'Zb$. It is also seen from the figure that $C'ZL$ is the angle N, because this angle and also NZH are both equal to 90° minus the arc $C'N$. Hence,

$$LZb = C'ZL - C'Zb = N - \gamma_2$$

This arc Lb is the hypothenuse of a spherical triangle whose base lies in the meridian passing through the equinoxial point L, and which is therefore the latitude of the point b. Likewise the sine of $(N - \gamma_2)$ is the hypothenuse of a plane triangle of which the base is $\sin (N - \gamma_2) \cos \delta'$. This shows the meaning of this factor in these formulæ. We can lay off the angle δ' from the point L, and a perpendicular from b let fall upon it gives the sine of the latitude. In the example this point falls between the two lines shown in the figure.

It is plainly evident that the formulæ give the southern limiting parallel, because the point lies in the fundamental plane and the construction is more easy in consequence. It is not so plainly seen that the formulæ give the northern parallel, because the point is in space and the construction more difficult to show the latitude itself.

In Fig. 28 we can see some of the other conditions attached to the formulæ when the whole shadow does not fall upon the earth. At

the north pole, the pole being elevated in the figure, an occultation is seen over and beyond the pole, and in fact all around it as far as the parallel $90 - \delta'$, so that $90°$ is the conventional northern limit here.

If the shadow should go off the earth at the south pole, which is depressed, the nearest parallel to the pole is numerically $90 - \delta$, but the formula $\delta - 90$ gives the parallel algebraically whether north or south. The shadow cannot reach further south than this, for that portion of the earth is below the horizon. The condition for the shadow falling partly off the earth is γ_1 or γ_2 greater than unity; this gives the limiting lines of the path, as above shown, falling beyond the earth, whose radius is unity.

In the elements of occultation, therefore, when we see δ' and y' (the motion north or south) having the *same* sign, we may expect to see $90°$ for some of the limiting parallels; but when δ' and y' have different signs, the shadow *may* go off the earth at the depressed pole, which will be known by examining δ' for the condition, $\varphi = 90 - \delta'$, or as we have in the *Nautical Almanac*, φ and δ, $\varphi + \delta = 90°$. If the shadow partly falls off the earth, y_0 is generally greater than unity; but if the path is much inclined (y' large), the limit is greater.

An occultation if shown in a chart like the eclipses, would be very similar to an eclipse, the rising and setting curve, northern and southern limits, outline, etc., as shown in Figs. 17 and 18, Plates VI. and VII., but there would be no central line or curves.

205. It is possible to find the limiting parallels graphically by the method of descriptive geometry. Fig. 29, Plate XI., shows the earth's sphere and path of the shadow, as in the previous figure. Consider a vertical plane passed through the line aa', the northern limit of which we want to find the limiting parallel. Revolve this plane toward the lower part of the drawing. The plane will cut from the earth a small circle, which is seen passing diametrically through the points a and a'. We now want the position of the north pole. In the revolution it will fall in the indefinite line PC, drawn perpendicular to the axis of revolution aa' and crossing it at D. In the previous figure we find the height of the pole above the fundamental plane to be PP'. Drawing the line $PE = PP'$ perpendicular to the line PC, the distance of the pole from the axis aa', which is a line in space before the revolution, is DE. An observer at a would see the pole revolve from E through an arc of $90°$ to F,

and the projection of its path on the plane of the paper is PC, C being found by the perpendicular FC, so that in the revolution of this portion of the sphere the small circle represents every portion of the earth over which the northern limiting line has passed. And C being the north pole, it is seen that the most northerly parallel of latitude touched by the northern limit of the path is that portion of this small circle which is *nearest to the pole.* Drawing from the centre of this circle, J, the line JC, it meets this circle at H and H', and the polar distance of the limiting parallel is the arc CH. This we must now find.

We will again revolve this section of the sphere about the line HH' (which is in the fundamental plane), these being the points in which this axis intersects the small circle. The highest point of the surface, which is over the centre, J, will fall in the line JI drawn perpendicular to HH', at a distance equal to that shown in Fig. 28, CC', since HH' and CC' in both figures are similar cords of the circle, and both lie in the fundamental plane. The centre of the sphere will fall in the line JI produced backward. The north pole falls also in a line perpendicular to the axis of revolution at a distance, $CL = FC$. The points H and H' remain fixed. The sphere can be drawn from the centre above found, and should pass through these four points, H, H', I, and L, the pole.

The sphere as now represented shows the north pole, L, being in the fundamental plane, the sphere projected upon a meridian passing through the point of tangency, H, of the path and limiting parallel, whose polar distance is consequently the arc LH.

The southern parallel is very easily shown, since it lies in the fundamental plane. When the sphere was revolved as above and the pole P fell at P', the point b of the southern parallel fell at b''. Draw through this point the line cc', and through the point Z the line ee', both perpendicular to $P'Z$; the former will be the parallel of latitude and the latter the equator in their revolved position, and the latitude ec or $e'c'$ can be measured by scale.

The occultations have generally taken up the entire time of one person throughout the year in the *Nautical Almanac* office, and sometimes two or three have been engaged upon the computations. The author has paid comparatively little attention to this division of the subject treated of in this work, his time being fully occupied with the eclipses and the computations for the planets Mercury, Venus, Mars, Jupiter, Saturn, Uranus, and Neptune.

206. *Criterion of visibility at any place,* as given in the *Almanac* (see page 576, for 1905).

(1) The limiting parallels must include the latitude of the place.

(2) The quantity $H - \lambda$ (in the *Almanac* notation, or $H - \omega$ of the present work), taken without regard to sign, must be less than the semidiurnal arc of the stars by at least one hour. Upon very rare occasions an emersion might be seen in the east, or an immersion in the west, when this difference is a few minutes less than an hour.

(3) The sun must not be much more than an hour above the horizon at the local mean time, $T - \lambda$ $[T - \omega]$, unless the star is bright enough to be seen in the daytime.

Fig. 28 shows how unsatisfactory are the limiting parallels in determining the locations on the earth where the occultation will be visible; in the space shown in the figure above the northern limit and the much larger space below the southern limit, the occultation is not visible at all. The visibility does not include more than one-third or possibly one-half of the earth's surface contained between the limiting parallels.

Several graphic methods have been devised and published for narrowing the zone of visibility, so as to avoid the tedious computation. The orthographic method described in Section XIX., the method by semidiameters, which is used here in Fig. 28, is suggested. This could be drawn on a much larger scale, and the axes marked into tenths or smaller divisions of the radius, especially the axis of X between the values 0.515 and 0.600, which are the extreme values of the coördinate x'; and Y from 0 to ± 0.200, the extreme values y'. These should be marked in both directions, and the divisions for y' carried beyond the sphere to ± 1.40 for y_0, which value has nearly been reached by some occultations.

The direction of the path can be shown by a thread laid over the proper values of x', y', taken from the *Almanac,* and a line parallel thereto drawn through the proper value of y_0 gives the path, which can be done without measuring. The limiting lines are distant from the path 0.2723, a constant.

The ellipse of the given place then is needed with hour angles marked on it. The location of the given place can be determined by its hour angle. The Washington hour angle and the time of conjunction are both given in the *Almanac.* The method is precisely that described in the section on Method by Semidiameters, Section XIX., which is illustrated in Fig. 21, Plate VIII.

The following is also suggested as a substitute for the limiting parallels: That instead of these, the *Nautical Almanac* might give the latitudes and longitudes of three or four points on each of the two limiting lines of the path, which, when plotted on a map, would show much more closely and clearly the regions in which the occultation is visible.

LIST OF TABLES

AND REFERENCES TO PAGES.

Table I.—Reduction of Time to Arc.

Hours.		Minutes.						Seconds.					
0^h	0°	0^m	0°	0′	30^m	7°	30′	0^s	0′	0″	30^s	7′	30″
1	15	1	0	15	31	7	45	1	0	15	31	7	45
2	30	2	0	30	32	8	0	2	0	30	32	8	0
3	45	3	0	45	33	8	15	3	0	45	33	8	15
4	60	4	1	0	34	8	30	4	1	0	34	8	30
5	75	5	1	15	35	8	45	5	1	15	35	8	45
6	90	6	1	30	36	9	0	6	1	30	36	9	0
7	105	7	1	45	37	9	15	7	1	45	37	9	15
8	120	8	2	00	38	9	30	8	2	0	38	9	30
9	135	9	2	15	39	9	45	9	2	15	39	9	45
10	150	10	2	30	40	10	0	10	2	30	40	10	0
11	165	11	2	45	41	10	15	11	2	45	41	10	15
12	180	12	3	0	42	10	30	12	3	0	42	10	30
13	195	13	3	15	43	10	45	13	3	15	43	10	45
14	210	14	3	30	44	11	0	14	3	30	44	11	0
15	225	15	3	45	45	11	15	15	3	45	45	11	15
16	240	16	4	0	46	11	30	16	4	0	46	11	30
17	255	17	4	15	47	11	45	17	4	15	47	11	45
18	270	18	4	30	48	12	0	18	4	30	48	12	0
19	285	19	4	45	49	12	15	19	4	45	49	12	15
20	300	20	5	0	50	12	30	20	5	0	50	12	30
21	315	21	5	15	51	12	45	21	5	15	51	12	45
22	330	22	5	30	52	13	0	22	5	30	52	13	0
23	345	23	5	45	53	13	15	23	5	45	53	13	15
24	360	24	6	0	54	13	30	24	6	0	54	13	30
		25	6	15	55	13	45	25	6	15	55	13	45
		26	6	30	56	14	0	26	6	30	56	14	0
		27	6	45	57	14	15	27	6	45	57	14	15
		28	7	0	58	14	30	28	7	0	58	14	30
		29	7	15	59	14	45	29	7	15	59	14	45
		30	7	30	60	15	0	30	7	30	60	15	0

Decimals.			
$0^s.00$	0″.00	$0^s.50$	7″.50
.01	0 .15	.51	7 .65
.02	0 .30	.52	7 .80
.03	0 .45	.53	7 .95
.04	0 .60	.54	8 .10
.05	0 .75	.55	8 .25
.06	0 .90	.56	8 .40
.07	1 .05	.57	8 .55
.08	1 .20	.58	8 .70
.09	1 .35	.59	8 .85
.10	1 .50	.60	9 .00
.11	1 .65	.61	9 .15
.12	1 .80	.62	9 .30
.13	1 .95	.63	9 .45
.14	2 .10	.64	9 .60
.15	2 .25	.65	9 .75
.16	2 .40	.66	9 .90
.17	2 .55	.67	10 .05
.18	2 .70	.68	10 .20
.19	2 .85	.69	10 .35
.20	3 .00	.70	10 .50
.21	3 .15	.71	10 .65
.22	3 .30	.72	10 .80
.23	3 .45	.73	10 .95
.24	3 .60	.74	11 .10
.25	3 .75	.75	11 .25
.26	3 .90	.76	11 .40
.27	4 .05	.77	11 .55
.28	4 .20	.78	11 .70
.29	4 .35	.79	11 .85
.30	4 .50	.80	12 .00
.31	4 .65	.81	12 .15
.32	4 .80	.82	12 .30
.33	4 .95	.83	12 .45
.34	5 .10	.84	12 .60
.35	5 .25	.85	12 .75
.36	5 .40	.86	12 .90
.37	5 .55	.87	13 .05
.38	5 .70	.88	13 .20
.39	5 .85	.89	13 .35
.40	6 .00	.90	13 .50
.41	6 .15	.91	13 .65
.42	6 .30	.92	13 .80
.43	6 .45	.93	13 .95
.44	6 .60	.94	14 .10
.45	6 .75	.95	14 .25
.46	6 .90	.96	14 .40
.47	7 .05	.97	14 .55
.48	7 .20	.98	14 .70
.49	7 .35	.99	14 .85
.50	7 .50	1 .00	15 .00

$0^s.000$	0″.000
1	0 .015
2	0 .030
3	0 .045
4	0 .060
5	0 .075
6	0 .090
7	0 .105
8	0 .120
9	0 .135
0.010	0 .150

Reduce $11^h\ 12^m\ 24^s.983$ to arc.

11^h	165°		
12^m	3	0	
24^s		6	0
.98			14.70
.003			.045

$11^h\ 12^m\ 24^s.983 = 168°\ 6'\ 14''.745$

Table II.—For Interpolating the Sun's Right Ascension and Declination, Log r' and Moon's Parallax for the Eclipse Hours.

Δ_2	1 23	2 22	3 21	4 20	5 19	6 18	7 17	8 16	9 15	10 14	11 13	12
0^h.00	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.05	.0010	.0019	.0027	.0035	.0041	.0047	.0052	.0056	.0059	.0061	.0062	.0062
.10	.0020	.0038	.0055	.0069	.0082	.0094	.0103	.0111	.0117	.0122	.0124	.0125
.15	.0030	.0057	.0082	.0104	.0124	.0141	.0155	.0167	.0176	.0182	.0186	.0187
.20	.0040	.0076	.0109	.0139	.0165	.0187	.0207	.0222	.0234	.0243	.0248	.0250
.25	.0050	.0095	.0137	.0173	.0206	.0234	.0258	.0278	.0293	.0304	.0310	.0312
.30	.0060	.0115	.0164	.0208	.0247	.0281	.0310	.0333	.0352	.0365	.0372	.0375
.35	.0070	.0134	.0191	.0243	.0288	.0328	.0362	.0389	.0410	.0425	.0434	.0437
.40	.0080	.0153	.0219	.0278	.0330	.0375	.0413	.0444	.0469	.0486	.0497	.0500
.45	.0090	.0172	.0246	.0312	.0371	.0422	.0465	.0500	.0527	.0547	.0559	.0562
.50	.0100	.0191	.0273	.0347	.0412	.0469	.0516	.0555	.0586	.0608	.0621	.0625
.55	.0110	.0210	.0301	.0382	.0453	.0516	.0568	.0611	.0645	.0668	.0683	.0687
.60	.0120	.0229	.0328	.0416	.0494	.0562	.0620	.0667	.0703	.0729	.0745	.0750
.65	.0130	.0248	.0356	.0451	.0536	.0609	.0671	.0722	.0762	.0790	.0807	.0812
.70	.0140	.0267	.0383	.0486	.0577	.0656	.0723	.0778	.0820	.0851	.0869	.0875
.75	.0150	.0280	.0410	.0520	.0618	.0703	.0775	.0833	.0879	.0911	.0931	.0937
.80	.0160	.0306	.0438	.0555	.0659	.0750	.0826	.0889	.0938	.0972	.0993	.1000
.85	.0170	.0325	.0465	.0590	.0700	.0797	.0878	.0944	.0996	.1033	.1055	.1062
.90	.0180	.0344	.0492	.0625	.0742	.0844	.0930	.1000	.1055	.1094	.1117	.1125
1.00	0.020	0.038	0.055	0.069	0.082	0.094	0.103	0.111	0.117	0.122	0.124	0.125
2.00	0.040	0.076	0.109	0.139	0.165	0.187	0.207	0.222	0.234	0.243	0.248	0.250
3.00	0.060	0.115	0.164	0.208	0.247	0.281	0.310	0.333	0.352	0.365	0.372	0.375
4.00	0.080	0.153	0.219	0.278	0.330	0.375	0.413	0.444	0.469	0.486	0.497	0.500
5.00	0.100	0.191	0.273	0.347	0.412	0.469	0.516	0.556	0.586	0.608	0.621	0.625
6.00	0.120	0.229	0.328	0.416	0.494	0.562	0.620	0.667	0.703	0.729	0.745	0.750
7.00	0.140	0.267	0.383	0.486	0.577	0.656	0.723	0.778	0.820	0.851	0.869	0.875
8.00	0.160	0.306	0.438	0.555	0.659	0.750	0.826	0.889	0.938	0.972	0.993	1.000
9.00	0.180	0.344	0.492	0.625	0.742	0.844	0.930	1.000	1.055	1.094	1.117	1.125
10.00	0.200	0.382	0.547	0.694	0.825	0.937	1.033	1.111	1.172	1.215	1.241	1.250
11.00	0.220	0.420	0.602	0.764	0.907	1.031	1.136	1.222	1.289	1.337	1.865	1.375
12.00	0.240	0.458	0.656	0.833	0.990	1.125	1.240	1.333	1.406	1.458	1.490	1.500
13.00	0.260	0.497	0.711	0.908	1.072	1.219	1.343	1.444	1.523	1.580	1.641	1.625
14.00	0.280	0.535	0.766	0.972	1.154	1.312	1.446	1.556	1.641	1.701	1.738	1.750
15.00	0.300	0.573	0.820	1.042	1.237	1.406	1.549	1.667	1.758	1.823	1.862	1.875
16.00	0.320	0.611	0.875	1.111	1.319	1.500	1.653	1.778	1.875	1.944	1.986	2.000
17.00	0.340	0.649	0.930	1.180	1.402	1.594	1.756	1.889	1.992	2.066	2.110	2.125
18.00	0.360	0.688	0.984	1.250	1.484	1.687	1.859	2.000	2.109	2.188	2.234	2.250
19.00	0.380	0.726	1.039	1.319	1.567	1.781	1.963	2.111	2.227	2.309	2.358	2.375
20.00	0.400	0.764	1.094	1.389	1.649	1.875	2.066	2.222	2.344	2.431	2.483	2.500
21.00	0.420	0.802	1.148	1.458	1.732	1.969	2.169	2.333	2.461	2.552	2.607	2.625
22.00	0.440	0.840	1.203	1.528	1.814	2.062	2.273	2.444	2.578	2.674	2.721	2.750
23.00	0.460	0.879	1.258	1.597	1.897	2.156	2.376	2.556	2.695	2.795	2.855	2.875
24.00	0.480	0.917	1.313	1.667	1.979	2.250	2.479	2.667	2.813	2.917	2.972	3.000
25.00	0.500	0.955	1.367	1.736	2.061	2.344	2.582	2.778	2.930	3.038	3.103	3.125
26.00	0.520	0.993	1.422	1.806	2.144	2.437	2.686	2.889	2.047	3.160	3.227	3.250
27.00	0.540	1.031	1.477	1.875	2.226	2.531	2.789	3.000	3.164	3.281	3.352	3.375
28.00	0.560	1.070	1.531	1.944	2.309	2.625	2.892	3.111	3.281	3.403	3.476	3.500
29.00	0.580	1.108	1.586	2.014	2.391	2.719	2.996	3.222	3.399	3.524	3.600	3.625
30.00	0.600	1.146	1.641	2.083	2.474	2.812	3.099	3.333	3.816	3.646	3.724	3.750

The upper part of this table is for the right ascension, and the lower part for the declination; but the parts are interchangeable by moving the decimal points.

When used for the moon's parallax, 1^h, 2^h, 3^h must be taken out in the table in the columns 2^h, 4^h, 6^h, etc.

Table III.—Factor for Interpolation giving Coefficient for Δ_2.

Δ_1 n		$\frac{n(n-1)}{1 \cdot 2}$	Δ_1 n
		18*	
0.00	—	0.00000	1.00
.01		0.00495	0.99
.02		0.00980	.98
.03		0.01455	.97
.04		0.01920	.96
.05	—	0.02375	.95
.06		0.02820	.94
.07		0.03255	.93
.08		0.03680	.92
.09		0.04095	.91
.10	—	0.04500	.90
.11		0.04895	.89
.12		0.05280	.88
.13		0.05655	.87
.14		0.06020	.86
.15	—	0.06375	.85
.16		0.06720	.84
.17		0.07055	.83
.18		0.07380	.82
.19		0.07695	.81
.20	—	0.08000	.80
.21		0.08295	.79
.22		0.08580	.78
.23		0.08855	.77
.24		0.09120	.76
.25	—	0.09375	.75
.26		0.09621	.74
.27		0.09855	.73
.28		0.10080	.72
.29		0.10295	.71
.30	—	0.10500	.70
.31		0.10695	.69
.32		0.10880	.68
.33		0.11055	.67
.34		0.11220	.66
.35	—	0.11375	.65
.36		0.11520	.64
.37		0.11655	.63
.38		0.11780	.62
.39		0.11895	.61
.40	—	0.12000	.60
.41		0.12095	.59
.42		0.12180	.58
.43		0.12255	.57
.44		0.12320	.56
.45	—	0.12375	.55
.46		0.12420	.54
.47		0.12455	.53
.48		0.12480	.52
.49		0.12495	.51
0.50	—	0.12500	0.50

Table IV.—Logarithm of the Earth's Radius.

Lat.	log ρ.		Lat.	log ρ.	
	9.999			9.999	
0°	* 000	1	45°	277	26
1	999	1	46	251	25
2	998	2	47	226	26
3	996	3	48	200	25
4	993	4	49	175	25
5	989	5	50	150	25
6	984	5	51	125	25
7	979	7	52	100	24
8	972	7	53	076	25
9	965	8	54	051	24
10	957	9	55	027	23
11	948	10	56	004	24
12	938	11	57	* 980	23
13	927	11	58	957	23
14	916	12	59	934	22
15	904	13	60	912	22
16	891	14	61	890	21
17	877	15	62	869	21
18	862	15	63	848	20
19	847	16	64	828	20
20	831	16	65	808	20
21	815	17	66	788	18
22	798	18	67	770	18
23	780	19	68	752	18
24	761	19	69	734	17
25	742	19	70	717	16
26	723	20	71	701	15
27	703	21	72	686	15
28	682	21	73	671	14
29	661	22	74	657	13
30	639	22	75	644	12
31	617	22	76	632	12
32	595	23	77	620	11
33	572	24	78	609	10
34	548	23	79	599	9
35	525	24	80	590	8
36	501	24	81	582	8
37	477	25	82	574	6
38	452	24	83	568	6
39	428	25	84	562	5
40	403	25	85	557	4
41	378	25	86	553	3
42	353	26	87	550	2
43	327	25	88	548	2
44	302	25	89	546	0
45	277	26	90	546	

* The argument of Table IV. is the geographical latitude.

Table V.

$\log \frac{1}{b}$		$\log \frac{1}{1-b}$
2.63828		0.0010000
	—431	
397		0100
	428	
2.62969		0200
	423	
546		0300
	419	
127		0400
	415	
2.61712		0500
	411	
301		0600
	407	
2.60894		0700
	404	
490		0800
	400	
090		0900
	396	
2.59694		1000
	392	
302		1100
	389	
2.58913		1200
	386	
527		1300
	382	
145		1400
	379	
2.57766		1500
	375	
391		1600
	373	
018		1700
	369	
2.56649		1800
	366	
283		1900
	—363	
2.55920		0.0012000

Table VI.

Mean Time.	Sidereal Equivalent in Arc.		
1ʰ	15°	2′	37″.85
2	30	4	55 .69
3	45	7	23 .54
4	60	9	51 .39
5	75	12	19 .24
6	90	14	47 .08
7	105	17	14 .93
8	120	19	42 .78
9	135	22	10 .62
10	150	24	38 .47
11	165	27	6 .32
12	180	29	34 .17
13	195	32	2 .01
14	210	34	29 .86
15	225	36	57 .71
16	240	39	25 .56
17	255	41	53 .40
18	270	44	21 .25
19	285	46	49 .10
20	300	49	16 .95
21	315	51	44 .80
22	330	54	12 .64
23	345	56	40 .49
24	360	59	8 .33

Table VII.

$\Delta\mu_1$.			$\log \mu_1'$.
14°	59′	40″	9.41780.8
		41	81.6
		42	82.4
		43	83.2
		44	84.0
14	59	45	9.41784.8
		46	85.6
		47	86.4
		48	87.2
		49	88.0
14	59	50	9.41788.8
		51	89.6
		52	90.4
		53	91.2
		54	92.0
14	59	55	9.41792.8
		56	93.6
		57	94.4
		58	95.2
		59	96.0
15	0	0	9.41796.8
		1	97.6
		2	98.4
		3	99.3
		4	9.41800.1
15	0	5	9.41800.9
		6	01.7
		7	02.5
		8	03.3
		9	04.1
15	0	10	9.41804.9
		11	05.7
		12	06.5
		13	07.3
		14	08.1
15	0	15	9.41808.9
		16	09.7
		17	10 6
15	0	18	9.41811.4

$\Delta\mu$.			$\log \Delta\mu_1$.
14°	59′	40″	1.17593
14	59	44	1.17595
14	59	50	1.17600
14	59	56	1.17605
15	0	1	1.17610
15	0	7	1.17615
15	0	14	1.17620
15	0	18	1.17623

Table VIII.

T.	Proportional Parts. μ_1.		
1ᵐ.	0°	15′	0″
2 .	0	30	0
3 .	0	45	0
4 .	1	0	0
5 .	1	15	0
6 .	1	30	0
7 .	1	45	0
8 .	2	0	0
9 .	2	15	0
10 .	2	30	0
0 .1	0	1	30
.2		3	0
.3		4	30
.4		6	0
.5		7	30
.6		9	0
.7		10	30
.8		12	0
.9		13	30
1 .0	0	15	0
0 .01	0	0′	9″
.02		0	18
.03		0	27
.04		0	36
.05		0	45
.06		0	54
.07		1	3
.08		1	12
.09		1	21
0 .10	0	1	30
0 .001	0	0″.9	
.002		1 .8	
.003		2 .7	
.004		3 .6	
.005		4 .5	
.006		5 .4	
.007		6 .3	
.008		7 .2	
.009		8 .1	
0 .010	0	9 .0	
0 .0001	0	0″.09	
.0002		0 .18	
.0003		0 .27	
.0004		0 .36	
.0005		0 .45	
.0006		0 .54	
.0007		0 .63	
.0008		0 .72	
.0009		0 .81	
0. 0010	0	0. 90	

Table IX.

Q.				log sin.	log cos.
0°	180°	360°		— ∞	0.00000
5	175	185	355	8.94030	9.99834
10	170	190	350	9.23967	9.99335
15	165	195	345	9.41300	9.98494
20	160	200	340	9.53405	9.97299
25	155	205	335	9.62595	9.95728
30	150	210	330	9.69897	9.93753
35	145	215	325	9.75859	9.91336
40	140	220	320	9.80807	9.88425
45	135	225	315	9.84949	9.84949
50	130	230	310	9.88425	9.80807
55	125	235	305	9.91336	9.75859
60	120	240	300	9.93753	9.69897
65	115	245	295	9.95728	9.62595
70	110	250	290	9.97299	9.53405
75	105	255	285	9.98494	9.41300
80	100	260	280	9.99335	9.23967
85	95	265	275	9.99834	8.94030
90		270		0.00000	∞

Table X.

$Q-\gamma$.				log ϵ. 7.6600		7.6675		7.6750	
0°	180°	360°		15′.7	0.1	16′.0	.1	16′.3	.1
5	175	185	355	15 .6	.1	15 .9	.2	16 .2	.2
10	170	190	350	15 .5	.3	15 .7	.3	16 .0	.3
15	165	195	345	15 .2	.4	15 .4	.3	15 .7	.4
20	160	200	340	14 .8	.6	15 .1	.6	15 .3	.5
25	155	205	335	14 .2	.6	14 .5	.7	14 .8	.7
30	150	210	330	13 .6	.7	13 .8	.7	14 .1	.8
35	145	215	325	12 .9	.9	13 .1	.8	13 .3	.8
40	140	220	320	12 .0	.9	12 .3	1.0	12 .5	1.0
45	135	225	315	11 .1	1.0	11 .3	1.0	11 .5	1.0
50	130	230	310	10 .1	1.0	10 .3	1.1	10 .5	1.2
55	125	235	305	9 .1	1.2	9 .2	1.2	9 .3	1.2
60	120	240	300	7 .9	1.2	8 .0	1.2	8 .1	1.3
65	115	245	295	6 .7	1.3	6 .8	1.4	6 .8	1.3
70	110	250	290	5 .4	1.3	5 .4	1.3	5 .5	1.3
75	105	255	285	4 .1	1.4	4 .1	1.4	4 .2	1.4
80	100	260	280	2 .7	1.3	2 .7	1.3	2 .8	1.4
85	95	265	275	1 .4	1.4	1 .4	1.4	1 .4	1.4
90	90	270	270	0 .0		0 .0		0 .0	

ϵ has the same sign as $\cos(Q-\gamma)$.

Table XI.

h.	log sin h.	log cos h.	cos h.	log sec h.
5	8.9403	9.9983	0.9924	0.0017
10	9.2397	9.9934	0.9698	0.0066
15	9.4130	9.9849	0.9330	0.0151
20	9.5341	9.9730	0.8830	0.0270
25	9.6259	9.9573	0.8214	0.0427
30	9.6990	9.9375	0.7500	0.0625
35	9.9586	9.9134	0.6710	0.0866
40	9.8081	9.8843	0.5868	0.1157
45	9.8495	9.8495	0 5000	0.1505
50	9.8843	9.8081	0.4132	0.1919
55	9.9134	9.7586	0.3290	0.2414
60	9.9375	9.6990	0.2500	0.3010
65	9.9573	9.6259	0.1786	0.3741
70	9.9730	9.5341	0.1170	0.4659
75	9.9849	9.4130	0.0670	0.5870
80	9.9934	9.2397	0.0302	0.7603
85	9.9983	8.9403	0.0076	1.0597
90	0.0000	— ∞	0.0000	∞

All the quantities in Table XI. are positive.

Table XII.

ϕ.	log F.		log G.		Angle of the Vertical.			
0°	0.00000	1	0.00295	1	—0′	0″	—2	0
5	01	3	294	3	2	0	1	55
10	04	6	291	6	3	55	1	49
15	10	7	285	7	5	44	1	39
20	17	9	278	9	7	23	1	25
25	26	11	269	11	8	48	1	9
30	37	11	258	11	9	57	0	51
35	48	13	247	13	10	48	0	32
40	61	13	234	13	11	20	—0	11
45	74	12	221	12	11	31	+0	10
50	86	13	209	13	11	21	0	31
55	99	12	196	12	10	50	0	51
60	111	10	184	10	9	59	1	9
65	121	9	174	9	8	50	1	25
70	130	8	165	8	7	25	1	39
75	138	5	157	5	5	46	1	49
80	143	3	152	3	3	57	1	57
85	146	1	149	2	2	0	+2	0
90	0.00147		0.00147		—0	0		

Table XIII., Giving ν' in Degrees and Decimal.

Top Argument, Natural Sine β. Side Argument, $\log \frac{f}{e}$.

	0.996	0.985	0.966	0.940	0.906	0.866	0.819	0.766	0.707	0.643	0.574	0.500	0.423	0.342	0.259	0.174	0.087	0.000
9.50	1.6	3.1	4.7	6.2	7.6	9.0	10.3	11.5	12.6	13.6	14.5	15.3	16.0	16.5	17.0	17.3	17.5	17.6
9.51	1.6	3.2	4.8	6.3	7.8	9.2	10.5	11.7	12.9	13.9	14.8	15.6	16.3	16.9	17.4	17.7	17.9	17.9
9.52	1.6	3.3	4.9	6.5	8.0	9.4	10.7	12.0	13.2	14.2	15.2	16.0	16.7	17.3	17.7	18.1	18.2	18.3
9.53	1.7	3.4	5.0	6.6	8.1	9.6	11.0	12.3	13.5	14.6	15.5	16.4	17.1	17.7	18.1	18.4	18.6	18.7
9.54	1.7	3.4	5.1	6.8	8.3	9.8	11.2	12.6	13.8	14.9	15.9	16.7	17.3	18.0	18.5	18.8	19.0	19.1
9.55	1.8	3.5	5.2	6.9	8.5	10.1	11.5	12.8	14.1	15.2	16.2	17.1	17.8	18.4	18.9	19.3	19.5	19.5
9.56	1.8	3.6	5.4	7.1	8.7	10.3	11.8	13.1	14.4	15.5	16.6	17.4	18.2	18.8	19.3	19.7	19.9	19.9
9.57	1.8	3.7	5.5	7.2	8.9	10.5	12.0	13.4	14.7	15.9	16.9	17.8	18.6	19.2	19.7	20.1	20.3	20.4
9.58	1.9	3.8	5.6	7.4	9.1	10.8	12.3	13.7	15.0	16.2	17.3	18.2	19.0	19.7	20.2	20.5	20.7	20.8
9.59	1.9	3.9	5.8	7.6	9.3	11.0	12.6	14.0	15.4	16.6	17.7	18.6	19.4	20.1	20.6	21.0	21.2	21.3
9.60	2.0	4.0	5.9	7.8	9.5	11.3	12.9	14.4	15.7	17.0	18.1	19.0	19.8	20.5	21.0	21.4	21.6	21.7
9.61	2.0	4.0	6.0	7.9	9.8	11.5	13.1	14.7	16.1	17.3	18.5	19.4	20.3	21.0	21.5	21.9	22.1	22.2
9.62	2.1	4.1	6.2	8.1	10.0	11.8	13.4	15.0	16.4	17.7	18.9	19.8	20.7	21.4	21.9	22.3	22.5	22.6
9.63	2.2	4.2	6.3	8.3	10.2	12.0	13.7	15.3	16.8	18.1	19.3	20.3	21.1	21.8	22.4	22.8	23.0	23.1
9.64	2.2	4.3	6.4	8.5	10.4	12.3	14.0	15.7	17.1	18.5	19.7	20.7	21.6	22.3	22.9	23.3	23.5	23.6
9.65	2.2	4.4	6.6	8.7	10.7	12.6	14.4	16.0	17.5	18.9	20.1	21.1	22.0	22.8	23.3	23.7	24.0	24.1
9.66	2.3	4.5	6.7	8.9	10.9	12.9	14.7	16.4	17.9	19.3	20.5	21.6	22.5	23.2	23.8	24.2	24.3	24.6
9.67	2.3	4.6	6.9	9.1	11.2	13.2	15.0	16.7	18.3	19.7	21.0	22.0	23.0	23.7	24.3	24.7	25.0	25.1
9.68	2.4	4.8	7.1	9.3	11.4	13.5	15.3	17.1	18.7	20.1	21.4	22.5	23.5	24.2	24.8	25.2	25.5	25.6
9.69	2.4	4.9	7.2	9.5	11.7	13.8	15.7	17.5	19.1	20.6	21.9	23.0	23.9	24.7	25.3	25.7	26.0	26.1
9.70	2.5	5.0	7.4	9.7	12.0	14.1	16.0	17.8	19.5	21.0	22.3	23.5	24.4	25.2	25.8	26.3	26.5	26.6
9.71	2.6	5.1	7.6	9.9	12.2	14.4	16.4	18.2	19.9	21.4	22.8	23.9	24.9	25.7	26.4	26.8	27.1	27.1
9.72	2.6	5.2	7.8	10.2	12.5	14.7	16.8	18.6	20.3	21.9	23.3	24.4	25.4	26.2	26.9	27.3	27.6	27.7
9.73	2.7	5.3	7.9	10.4	12.8	15.0	17.1	19.0	20.7	22.4	23.7	24.9	26.0	26.8	27.4	27.9	28.1	28.2
9.74	2.8	5.5	8.1	10.6	13.1	15.4	17.5	19.4	21.2	22.8	24.2	25.6	26.5	27.3	28.0	28.4	28.7	28.8
9.75	2.8	5.6	8.3	10.9	13.4	15.7	17.9	19.9	21.7	23.3	24.7	26.0	27.0	27.9	28.5	29.0	29.2	29.3
9.76	2.9	5.7	8.5	11.1	13.7	16.1	18.3	20.3	22.1	23.8	25.2	26.5	27.5	28.4	29.1	29.5	29.8	29.9
9.77	3.0	5.9	8.7	11.4	14.0	16.4	18.7	20.7	22.6	24.3	25.7	27.0	28.1	29.0	29.6	30.1	30.4	30.5
9.78	3.0	6.0	8.9	11.6	14.3	16.8	19.1	21.2	23.1	24.8	26.3	27.5	28.6	29.5	30.2	30.7	31.0	31.1
9.79	3.1	6.1	9.1	11.9	14.6	17.1	19.5	21.6	23.6	25.3	26.8	28.1	29.2	30.1	30.8	31.3	31.5	31.8
9.80	3.2	6.3	9.3	12.2	14.9	17.5	19.9	22.1	24.0	25.8	27.3	28.6	29.8	30.7	31.4	31.8	32.1	32.4
9.81	3.2	6.4	9.5	12.4	15.3	17.9	20.3	22.5	24.5	26.3	27.9	29.2	30.3	31.2	31.9	32.4	32.7	33.0
9.82	3.3	6.6	9.7	12.7	15.6	18.3	20.7	23.0	25.0	26.8	28.4	29.8	30.9	31.8	32.5	33.0	33.3	33.4
9.83	3.4	6.7	9.9	13.0	15.9	18.7	21.2	23.5	25.6	27.4	29.0	30.3	31.5	32.4	33.1	33.6	34.0	34.1
9.84	3.5	6.9	10.2	13.3	16.3	19.1	21.6	24.0	26.1	27.9	29.5	30.9	32.1	33.0	33.7	34.3	34.6	34.7
9.85	3.5	7.0	10.4	13.6	16.6	19.5	22.1	24.5	26.6	28.5	30.1	31.5	32.7	33.6	34.4	34.9	35.2	35.3

Table XIV.—Part I., Giving Q in Degrees and Decimal.

Top Argument, ν' from Table XIII. Side Argument, E.

	0.0	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	11.0	12.0	13.0	14.0	15.0	16.0	17.0
0°	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
1	1.0	1.0	1.0	1.1	1.1	1.1	1.1	1.1	1.2	1.2	1.2	1.2	1.3	1.3	1.3	1.4	1.4	1.6
2	2.0	2.0	2.1	2.1	2.1	2.2	2.2	2.3	2.3	2.4	2.4	2.5	2.5	2.6	2.7	2.7	2.8	2.9
3	3.0	3.1	3.1	3.2	3.2	3.3	3.3	3.4	3.5	3.6	3.6	3.7	3.8	3.9	4.0	4.1	4.2	4.3
4	4.0	4.1	4.1	4.2	4.3	4.4	4.5	4.6	4.7	4.8	4.9	5.0	5.1	5.2	5.3	5.5	5.6	5.8
5	5.0	5.1	5.2	5.3	5.4	5.5	5.6	5.7	5.8	5.9	6.1	6.2	6.3	6.5	6.6	6.8	7.0	7.2
6	6.0	6.1	6.2	6.3	6.5	6.6	6.7	6.8	7.0	7.1	7.3	7.4	7.6	7.8	8.0	8.2	8.4	8.6
7	7.0	7.1	7.3	7.4	7.5	7.7	7.8	8.0	8.1	8.3	8.5	8.7	8.9	9.1	9.3	9.5	9.8	10.1
8	8.0	8.1	8.3	8.4	8.6	8.8	8.9	9.1	9.3	9.5	9.7	9.9	10.1	10.4	10.6	10.9	11.2	11.5
9	9.0	9.2	9.3	9.5	9.7	9.9	10.0	10.2	10.6	10.7	10.9	11.1	11.4	11.7	11.9	12.2	12.6	12.9
10	10.0	10.2	10.4	10.5	10.7	10.9	11.2	11.4	11.6	11.9	12.1	12.4	12.7	13.0	13.3	13.6	14.0	14.3
11	11.0	11.2	11.4	11.6	11.8	12.0	12.3	12.5	12.8	13.0	13.3	13.6	13.9	14.3	14.6	15.0	15.3	15.8
12	12.0	12.2	12.4	12.6	12.9	13.1	13.4	13.6	13.9	14.2	14.5	14.8	15.2	15.5	15.9	16.3	16.7	17.2
13	13.0	13.2	13.5	13.7	14.0	14.2	14.5	14.8	15.1	15.4	15.7	16.1	16.5	16.8	17.2	17.7	18.1	18.6
14	14.0	14.2	14.5	14.7	15.0	15.3	15.6	15.9	16.3	16.6	16.9	17.3	17.7	18.1	18.5	19.0	19.5	20.0
15	15.0	15.3	15.5	15.8	16.1	16.4	16.7	17.0	17.4	17.8	18.1	18.5	19.0	19.4	19.8	20.3	20.8	21.4
16	16.0	16.3	16.6	16.9	17.2	17.5	17.8	18.2	18.6	18.9	19.3	19.8	20.2	20.7	21.1	21.7	22.2	22.8
17	17.0	17.3	17.6	17.9	18.3	18.6	18.9	19.3	19.7	20.1	20.5	21.0	21.5	21.9	22.4	23.0	23.6	24.2
18	18.0	18.3	18.6	19.0	19.3	19.7	20.1	20.4	20.9	21.3	21.7	22.2	22.7	23.2	23.7	24.3	24.9	25.6
19	19.0	19.3	19.7	20.0	20.4	20.8	21.2	21.6	22.0	22.5	22.9	23.4	23.9	24.5	25.0	25.7	26.3	27.0
20	20.0	20.3	20.7	21.1	21.5	21.9	22.3	22.7	23.2	23.6	24.1	24.6	25.2	25.7	26.3	27.0	27.6	28.3
21	21.0	21.4	21.7	22.1	22.5	23.0	23.4	23.8	24.3	24.8	25.3	25.9	26.4	27.0	27.6	28.3	29.0	29.7
22	22.0	22.4	22.8	23.2	23.6	24.0	24.5	25.0	25.5	26.0	26.5	27.1	27.6	28.3	28.9	29.6	30.3	31.1
23	23.0	23.4	23.8	24.2	24.7	25.1	25.6	26.1	26.6	27.1	27.7	28.3	28.9	29.5	30.2	30.9	31.6	32.4
24	24.0	24.4	24.8	25.3	25.7	26.2	26.7	27.2	27.8	28.3	28.9	29.5	30.1	30.8	31.5	32.2	33.0	33.8
25	25.0	25.4	25.9	26.3	26.8	27.3	27.8	28.3	28.9	29.5	30.1	30.7	31.3	32.0	32.7	33.5	34.3	35.1
26	26.0	26.4	26.9	27.4	27.9	28.4	28.9	29.5	30.0	30.6	31.2	31.9	32.6	33.3	34.0	34.8	35.6	36.5
27	27.0	27.5	27.9	28.4	28.9	29.5	30.0	30.6	31.2	31.8	32.4	33.1	33.8	34.5	35.3	36.1	36.9	37.8
28	28.0	28.5	29.0	29.5	30.0	30.5	31.1	31.7	32.3	32.9	33.6	34.3	35.0	35.7	36.5	37.3	38.2	39.1
29	29.0	29.5	30.0	30.5	31.1	31.6	32.2	32.8	33.4	34.1	34.7	35.5	36.2	37.0	37.8	38.6	39.5	40.4
30	30.0	30.5	31.0	31.6	32.1	32.7	33.3	33.9	34.6	35.2	35.9	36.7	37.4	38.2	39.0	39.9	40.8	41.7
31	31.0	31.5	32.1	32.6	33.2	33.8	34.4	35.0	35.7	36.4	37.1	37.8	38.6	39.4	40.3	41.2	42.1	43.0
32	32.0	32.5	33.1	33.6	34.2	34.9	35.5	36.1	36.8	37.5	38.3	39.0	39.8	40.6	41.5	42.4	43.3	44.3
33	33.0	33.6	34.1	34.7	35.3	35.9	36.6	37.2	38.0	38.7	39.4	40.2	41.0	41.9	42.7	43.7	44.6	45.6
34	34.0	34.6	35.2	35.7	36.4	37.0	37.7	38.3	39.1	39.8	40.6	41.4	42.2	43.1	44.0	44.9	45.9	46.9
35	35.0	35.6	36.2	36.8	37.4	38.1	38.8	39.5	40.2	40.9	41.7	42.5	43.4	44.3	45.2	46.1	47.1	48.2

Table XIV.—Part II., Giving Q in Degrees and Decimal.

Top argument, ν' from Table XIII. Side argument, E.

	18.0	19.0	20.0	21.0	22.0	23.0	24.0	25.0	26.0	27.0	28.0	29.0	30.0	31.0	32.0	33.0	34.0	35.0
0°	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
1	1.5	1.5	1.6	1.6	1.7	1.7	1.8	1.9	2.0	2.0	2.1	2.2	2.4	2.5	2.7	2.9	3.1	3.4
2	3.0	3.0	3.1	3.3	3.4	3.5	3.6	3.7	3.9	4.1	4.3	4.3	4.7	5.0	5.4	5.7	6.2	6.7
3	4.4	4.6	4.7	4.9	5.0	5.2	5.4	5.6	5.9	6.1	6.4	6.7	7.1	7.6	8.0	8.5	9.2	10.0
4	5.9	6.1	6.2	6.5	6.7	6.9	7.2	7.5	7.8	8.2	8.5	8.9	9.4	10.0	10.6	11.3	12.2	13.2
5	7.4	7.6	7.8	8.1	8.4	8.7	9.0	9.3	9.7	10.2	10.6	11.1	11.7	12.4	13.2	14.1	15.1	16.4
6	8.9	9.1	9.4	9.7	10.0	10.4	10.8	11.2	11.7	12.2	12.7	13.3	14.0	14.9	15.8	16.8	18.0	19.5
7	10.3	10.6	11.0	11.3	11.7	12.1	12.6	13.0	13.6	14.2	14.8	15.5	16.3	17.3	18.3	19.5	20.9	22.6
8	11.8	12.1	12.5	12.9	13.3	13.8	14.3	14.8	15.5	16.1	16.9	17.7	18.6	19.6	20.8	22.2	23.7	25.6
9	13.3	13.7	14.1	14.5	15.0	15.5	16.1	16.7	17.4	18.1	18.9	19.8	20.8	22.0	23.3	24.9	26.5	28.5
10	14.7	15.2	15.6	16.1	16.6	17.2	17.8	18.5	19.2	20.1	21.0	22.0	23.1	24.3	25.7	27.4	29.2	31.4
11	16.2	16.7	17.2	17.7	18.3	18.9	19.6	20.3	21.1	22.0	22.9	24.1	25.3	26.6	28.1	29.9	31.9	34.2
12	17.6	18.1	18.7	19.3	19.9	20.6	21.3	22.1	23.0	23.9	24.9	26.1	27.4	28.9	30.5	32.4	34.5	37.0
13	19.1	19.6	20.2	20.8	21.5	22.2	23.0	23.9	24.8	25.8	26.9	28.2	29.5	31.1	32.7	34.8	36.9	39.8
14	20.5	21.1	21.7	22.4	23.1	23.9	24.7	25.6	26.6	27.7	28.8	30.2	31.6	33.2	35.0	37.1	39.3	42.5
15	22.0	22.6	23.3	24.0	24.7	25.5	26.4	27.4	28.4	29.6	30.8	32.2	33.7	35.3	37.2	39.3	41.6	
16	23.4	24.1	24.8	25.5	26.3	27.2	28.1	29.1	30.2	31.4	32.7	34.1	35.7	37.4	39.3	41.4		
17	24.8	25.5	26.3	27.1	27.9	28.8	29.8	30.8	32.0	33.2	34.6	36.0	37.6	39.5	41.4			
18	26.2	27.0	27.8	28.6	29.4	30.4	31.4	32.5	33.7	35.0	36.4	37.9	39.6	41.5				
19	27.7	28.4	29.2	30.1	31.0	32.0	33.1	34.2	35.4	36.7	38.2	39.7	41.5					
20	29.1	29.9	30.7	31.6	32.5	33.6	34.7	35.8	37.1	38.5	40.0	41.6						
21	30.5	31.3	32.2	33.1	34.1	35.1	36.2	37.5	38.8	40.2	42.0							
22	31.9	32.7	33.6	34.6	35.6	36.7	37.8	39.1	40.5	42.0								
23	33.2	34.1	35.1	36.0	37.1	38.2	39.3	40.7	42.1									
24	34.6	35.5	36.5	37.5	38.6	39.7	40.9	42.1										
25	36.0	36.9	37.9	39.0	40.1	41.2	42.5											
26	37.4	38.3	39.3	40.4	41.6	42.8	44.0											
27	38.7	39.7	40.7	41.8	43.0	44.2												
28	40.1	41.1	42.1	43.2	44.4	45.6												
29	41.4	42.4	43.5	44.6	45.8													
30	42.7	43.8	44.9	46.1	47.3													
31	44.1	45.1	46.2	47.4														
32	45.4	46.5	47.6	48.8														
33	46.7	47.8	48.9	50.0														
34	48.0	49.1	50.3															
35	49.2	50.4	51.0															

Table XV.—WHOLE NUMBER OF ECLIPSES IN ONE SAROS, 1889–1906.

Year.	Total Species.		Annular Species.		Lunar Eclipses.		In one Year.
	Total.	Partial.	Annular.	Partial.	Total.	Partial.	
1889	2		1			2	5
1890	—		2			1	3
1891	—		1	1	2		4
1892	1			1	1	1	4
1893	1		1		—		2
1894	1		1			2	4
1895		2		1	2		5
1896	1		1			2	4
1897	—	—	2		—		2
1898	1	1	1		1	2	6
1899		1	1	1	1	1	5
1900	1		1			1	3
1901	1		1			1	3
1902		2		1	2		5
1903	1		1			2	4
1904	1		1		—		2
1905	1		1			2	4
1906	—			3	2		5
Species in one Saros.	12	6	16	8	11	17	70

NOTE.—The dash denotes no eclipse of the species during the year. Lunar Appulse. 1890, and 1901.

THE APPROXIMATE LOCATIONS OF FUTURE ECLIPSES.

This may be done with little or no trouble as follows:

Take the eclipse 1904, Sept. 9, ☌ $8^h\ 49^m$. Applying the Saros on the next page in which we must take 11 days, we have for the date and approximate time of conjunction 1922, Sept. 20, $16^h\ 31^m$.

The eclipse At Noon, Art. 114, occurred in longitude +133 5 W., and latitude —4° 35′ south. During the $7^h\ 42^m$ of the Saros the earth will revolve through 115.6 degrees (see next page). The Eclipse *At Noon* will be in longitude +249 W., or 111 East. By the table on the next page, column 6, the series is seen to be moving South. This motion averages about 5° at each appearance, so that the eclipse At Noon will be in latitude —9° 35′ South. In Art. 45 we ascertained that the inclination of the path is *increasing* slightly, which will cause the beginning to fall only 2° or 3° south, while the ending will be 7° or 8° south of 1904.

By the present table, column 5, we see that the direction of the path is south, and in column 3 that the duration in column 2 is *decreasing*, so that it will be somewhat less than $6^m\ 24^s$ at its greatest appearance. These results are only approximate, for mean values alone can be used.

Table XVI.—List of All Total Eclipses in one Saros, 1889 to 1906.

Date.	Duration.	Increasing or Decreasing.	At Noon. ϕ ω	Direction of Path.	Motion of Series.
	m s		° °		
1889, Jan. 1	2 15	D	+37 138 W.	*N* slightly.	*N*
1889, Dec. 21	4 15	D	−14 73	*S*	*N*
1892, April 26	4 17	D	−64 139	*N*	*S*
1893, April 16	4 46	D	− 1 37	*N*	*S*
1894, Sept. 28	0 11	I	−34 274	*S*	*N*
1896, Aug. 8	2 42	D	+65 248	*S*	*N*
1898, Jan. 21	2 19	I	+13 291	*N*	*S*
1900, May 28	2 9	D	+45 45	*N*	*N*
1901, May 17	6 27	I	− 2 263	*N*	*N*
1903, Sept. 20	2 15	D	−70 259	*S*	*S*
1904, Sept. 9	6 24	D	− 5 133	*S*	*S*
1905, Aug. 29	3 45	I	+46 12	*S*	*S*
1906. All the Solar Eclipses will be Partial.					

List of All Annular Eclipses in one Saros, 1889 to 1906.

Date.	Duration.	Increasing or Decreasing.	At Noon. ϕ ω	Direction of Path.	Motion of Series.
	m s		° °		
1889, June 27	7 23	D	−10 313	*N*	*S*
1890, June 16	4 12	I	−37 330	*N*	*S*
1890, Dec. 11	0 17	I	−54 230	*S*	*N*
1891, June 6	Not given.	I	+70 250	*N*	*S*
1893, Oct. 9	3 46	D	+13 126	*S*	*N*
1894, April 5	0 1	I	+47 246	*N*	*S*
1896, Feb. 13	Not given.	D	S. None.	*N*	*S*
1897, Feb. 1	2 38	D	−29 118	*N*	*S*
1897, July 29	1 6	I	+15 58	*S*	*N*
1898, July 18	6 13	I	−43 120	*S*	*N*
1899, Dec. 2	Not given.	D	−88 198	*S*	*S*
1900, Nov. 21	0 44	D	−33 294	*S*	*S*
1901, Nov. 10	11 3	I	+12 294	*S*	*S*
1903, Mar. 28	1 49	D	+65 210	*N*	*N*
1904, Mar. 16	8 3	D	+ 6 264	*N*	*N*
1905, Mar. 5	7 58	I	−43 250	*N*	*N*

The Saros 6585.321222 days, or

$$18^{y}\ {}^{10}_{11}{}^{d}\ 7^{h}\ 42^{m}\ 33.6^{s}, \text{ if Feb. 29 intervenes} \begin{cases} 5 \text{ times.} \\ 4 \text{ times.} \end{cases}$$

Revolution of the earth in $7^{h}\ 42^{m}\ 33.6^{s} = 115°\ 6'$ west.

BY THE SAME AUTHOR

TREATISE

ON THE

PROJECTION OF THE SPHERE

WITH PRECEPTS AND TABLES

FOR FACILITATING THE

ORTHOGRAPHIC AND STEREOGRAPHIC PROJECTIONS

OF ANY PORTION OF THE EARTH'S SURFACE

BY ROBERDEAU BUCHANAN, S. B.

Assistant in the Office of the American Ephemeris and Nautical Almanac

AT

WASHINGTON, D. C., 1890

With plates, pp. 47

In this work the method given, with formulæ and tables, was devised by the author for his own use, and has been used by him yearly in projecting solar eclipses and in connection with other duties in the office of the American Ephemeris and Nautical Almanac.

PRIVATELY PRINTED. PRICE, 75 CENTS

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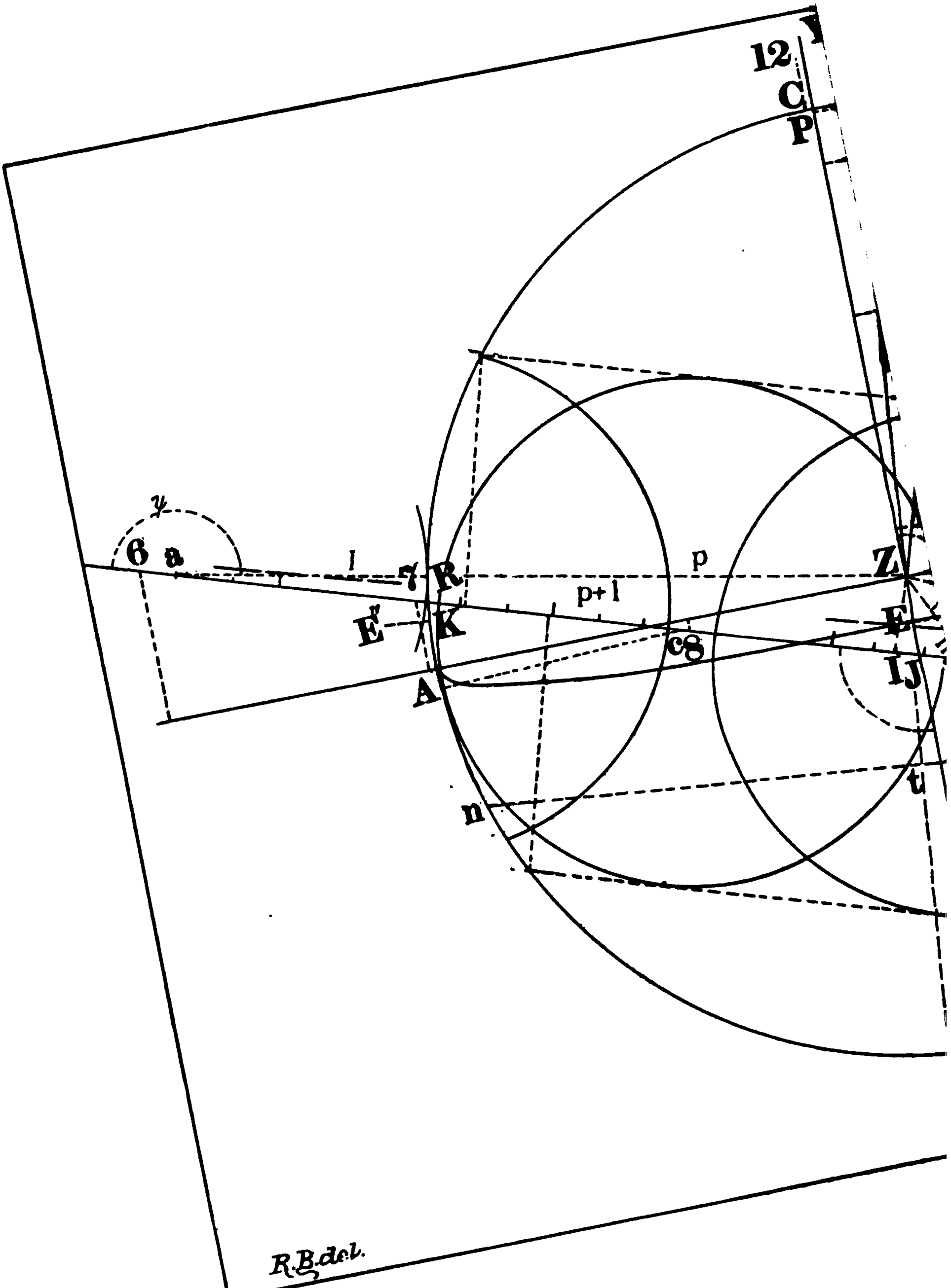
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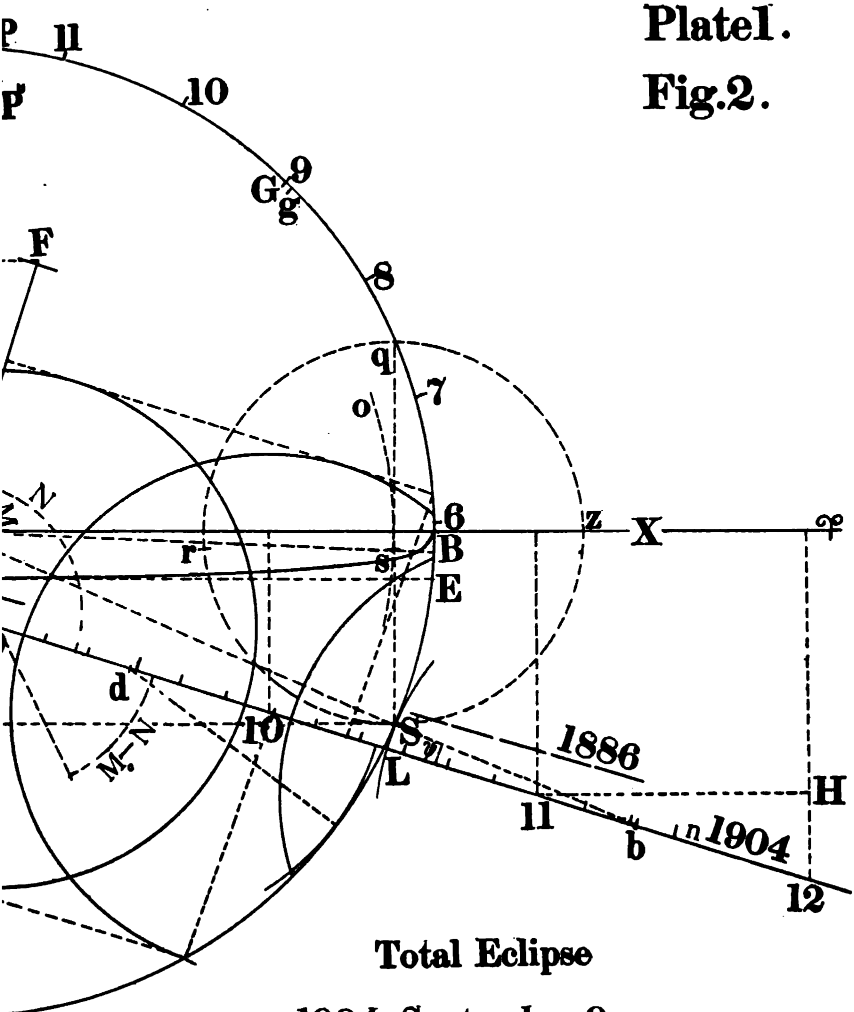
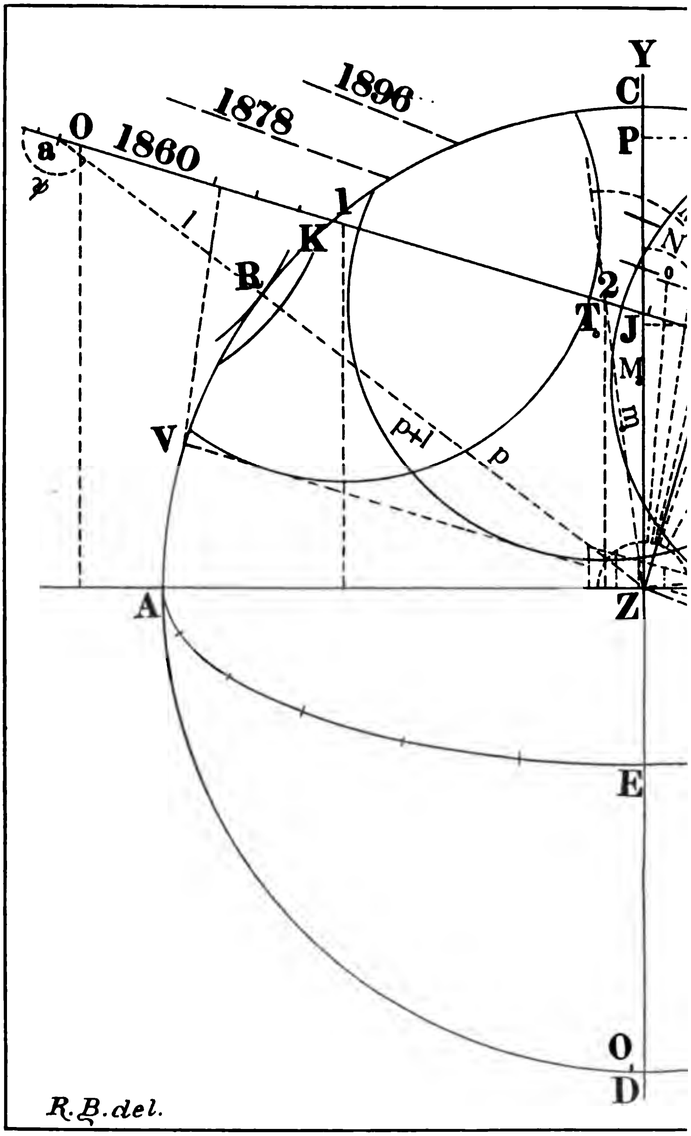
Plate I.

Fig. 2.



Total Eclipse

1904, September 9.



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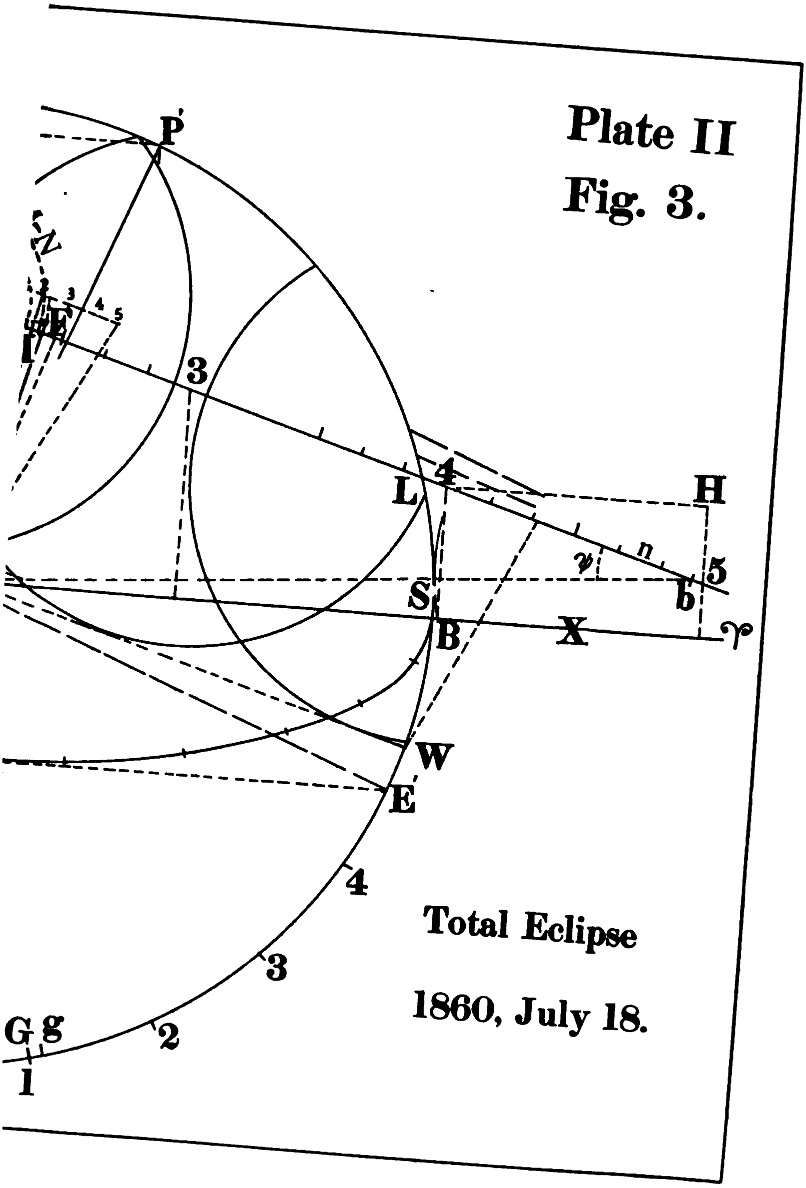

Plate II

Fig. 3.

Total Eclipse

1860, July 18.

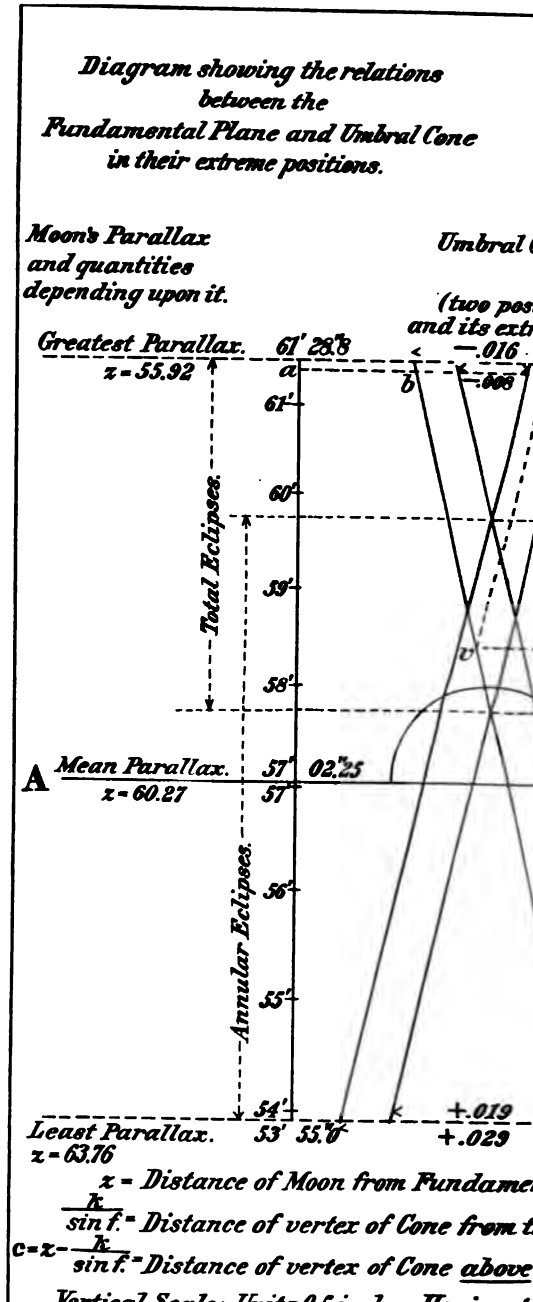

Diagram showing the relations between the Fundamental Plane and Umbral Cone in their extreme positions.
Moon's Parallax and quantities depending upon it.
Umbral C
(two posi
and its extr
Greatest Parallax. 61' 28.8"
z = 55.92
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60'
59'
58'
57'
56'
55'
54'
Total Eclipses.
Annular Eclipses.
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−.008
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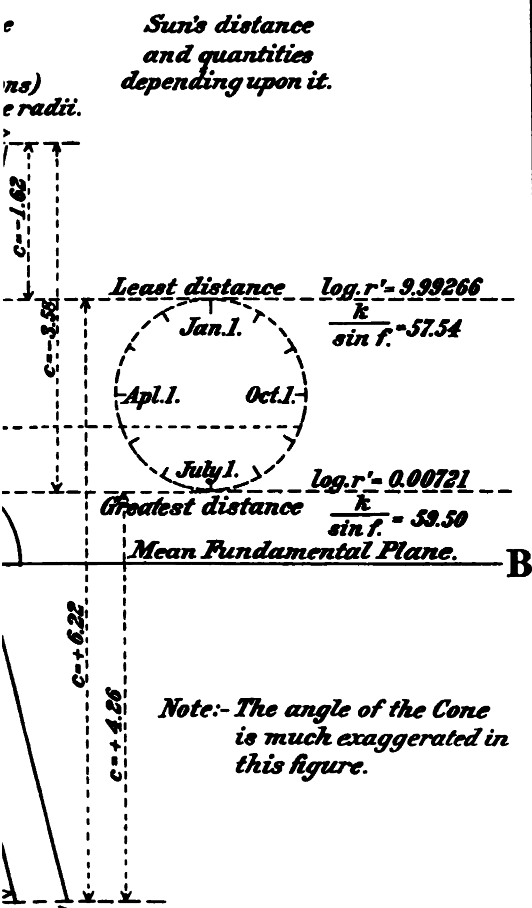
Mean Parallax. 57' 02.25"
z = 60.27
+.019
+.029
Least Parallax. 53' 55.0"
z = 63.76
z = Distance of Moon from Fundamen
k/sin f = Distance of vertex of Cone from th
c = z − k/sin f = Distance of vertex of Cone above
Vertical Scale: Unit = 0.5 inch. Horizonta

Plate III Fig. 4.

The Umbral Cone.



l Plane.

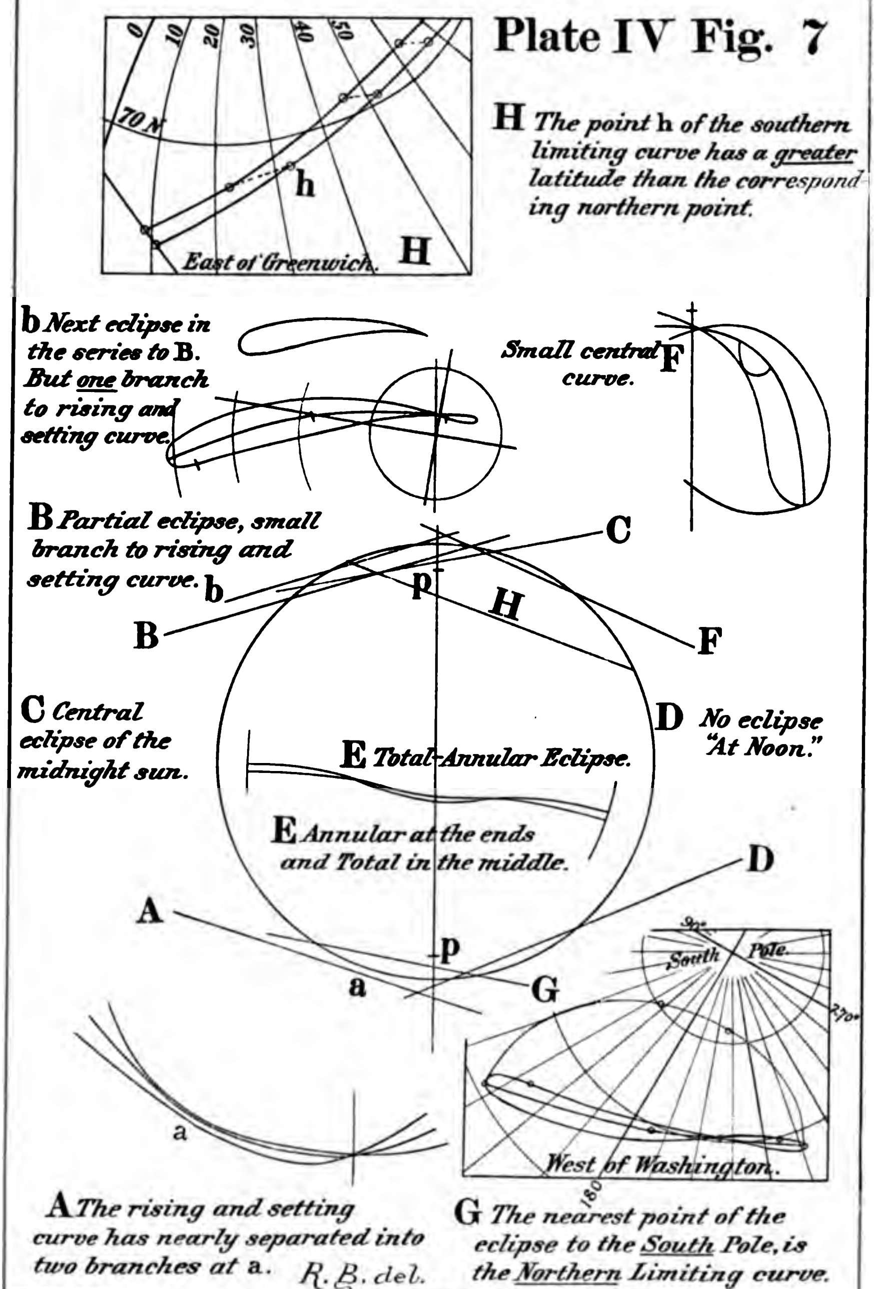
Moon.

ndamental Plane.

Scale: 0.01 = 0.5 inch.

R. B. del.

Peculiarities of Eclipses



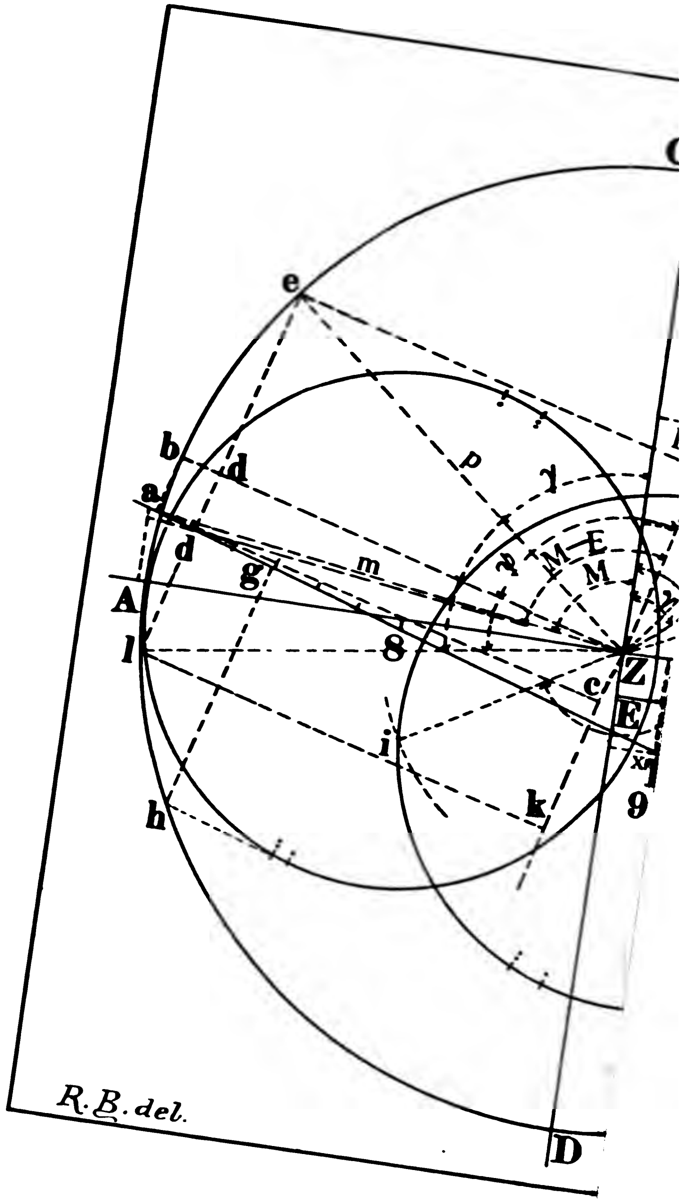

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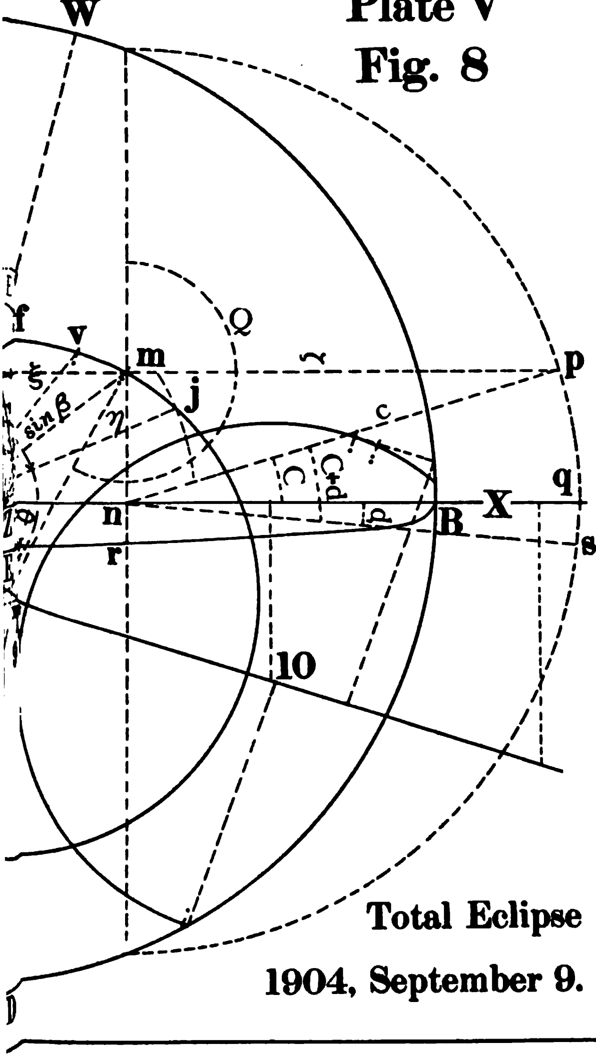
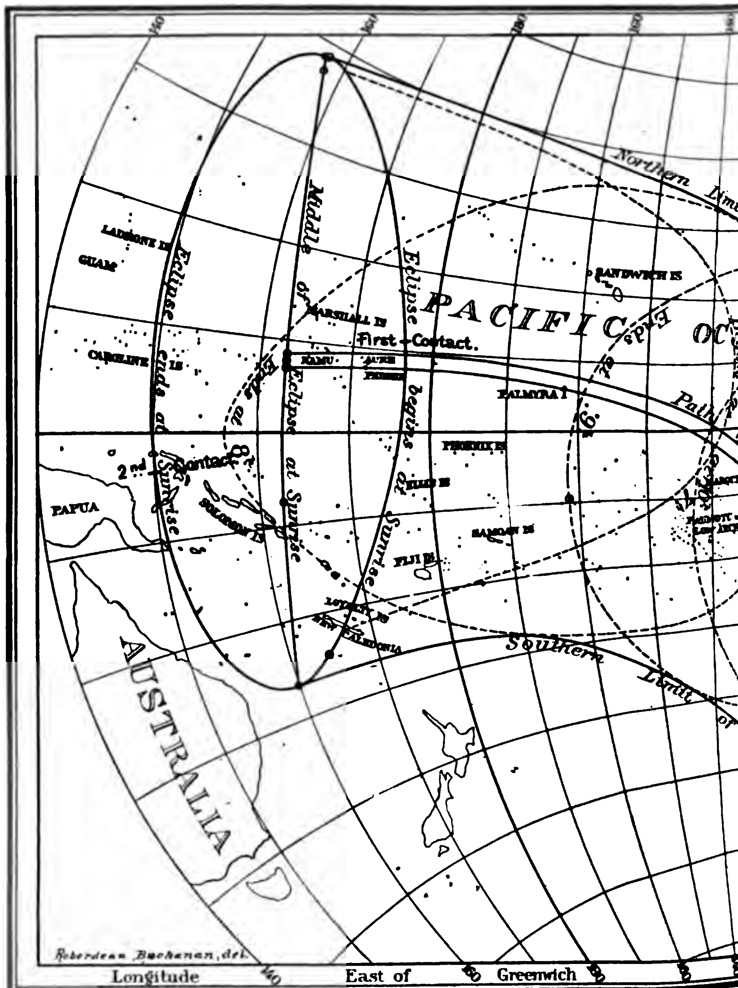
Plate V
Fig. 8



Total Eclipse

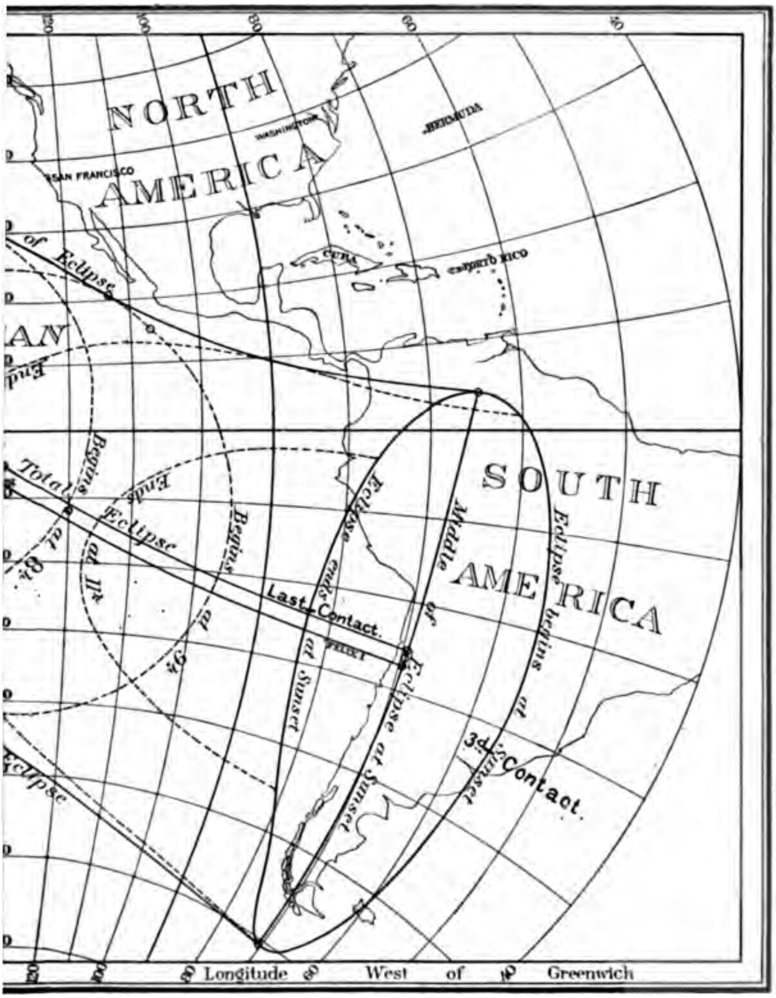
1904, September 9.

TOTAL ECLIPSE OF



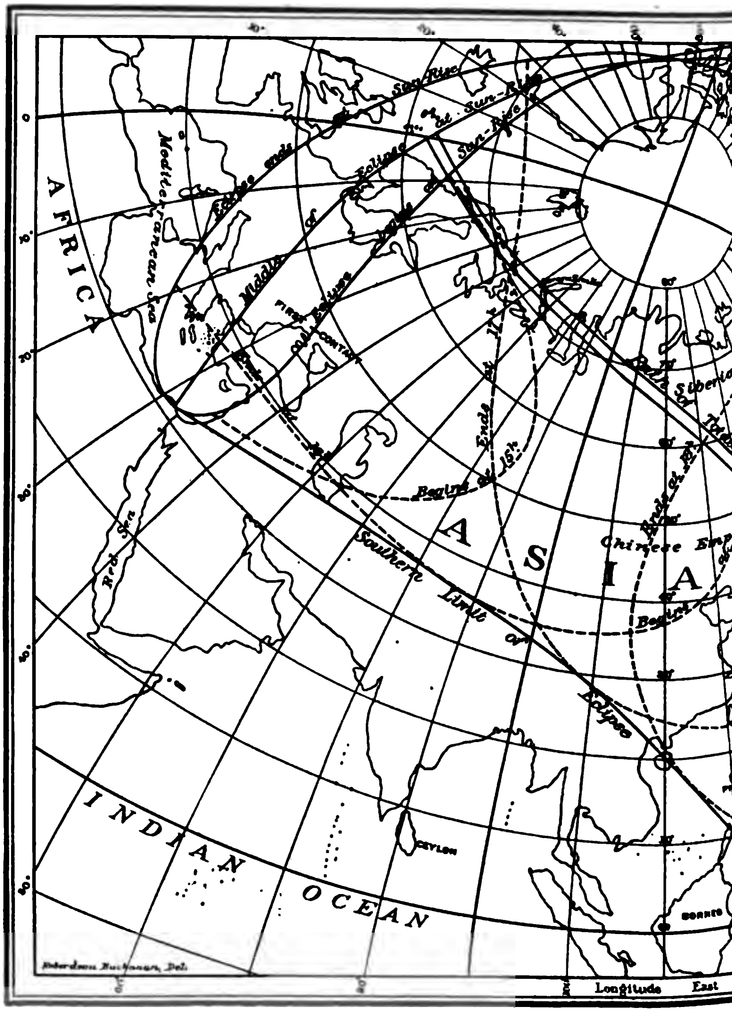
Note: *The hours of beginning and ending*

EPTEMBER 9TH 1904 **Plate VI, Fig. 17.**

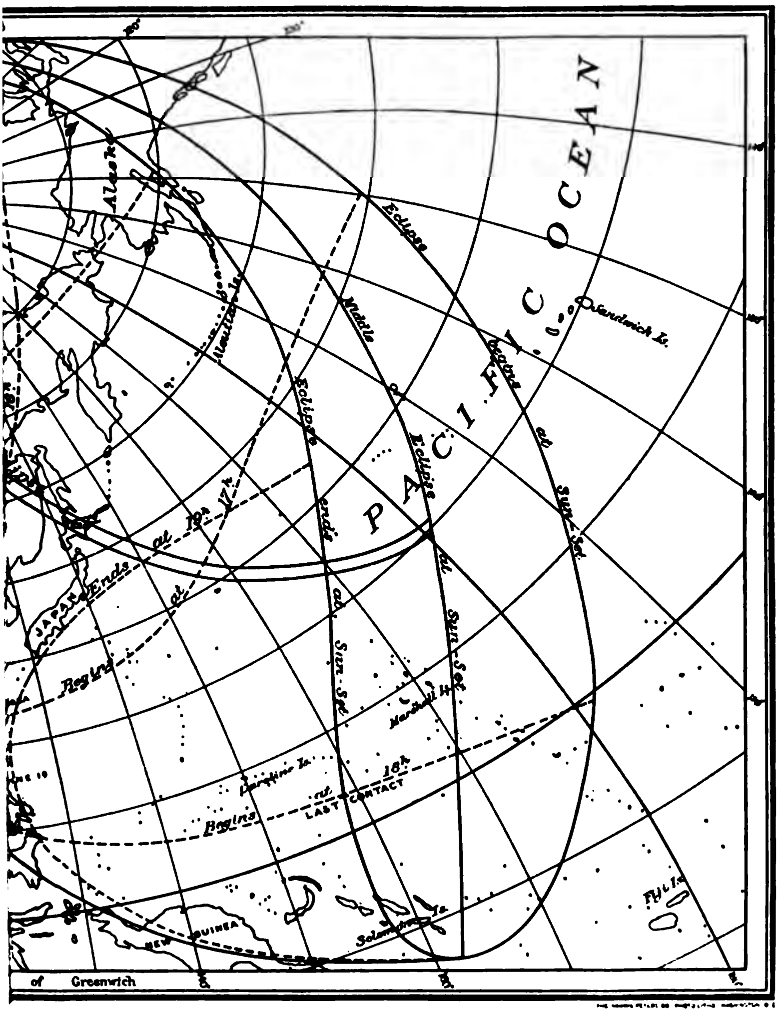


are expressed in Greenwich Mean Time.

TOTAL ECLIPSE



OF AUGUST 8TH 1896. Plate VII, Fig. 18.



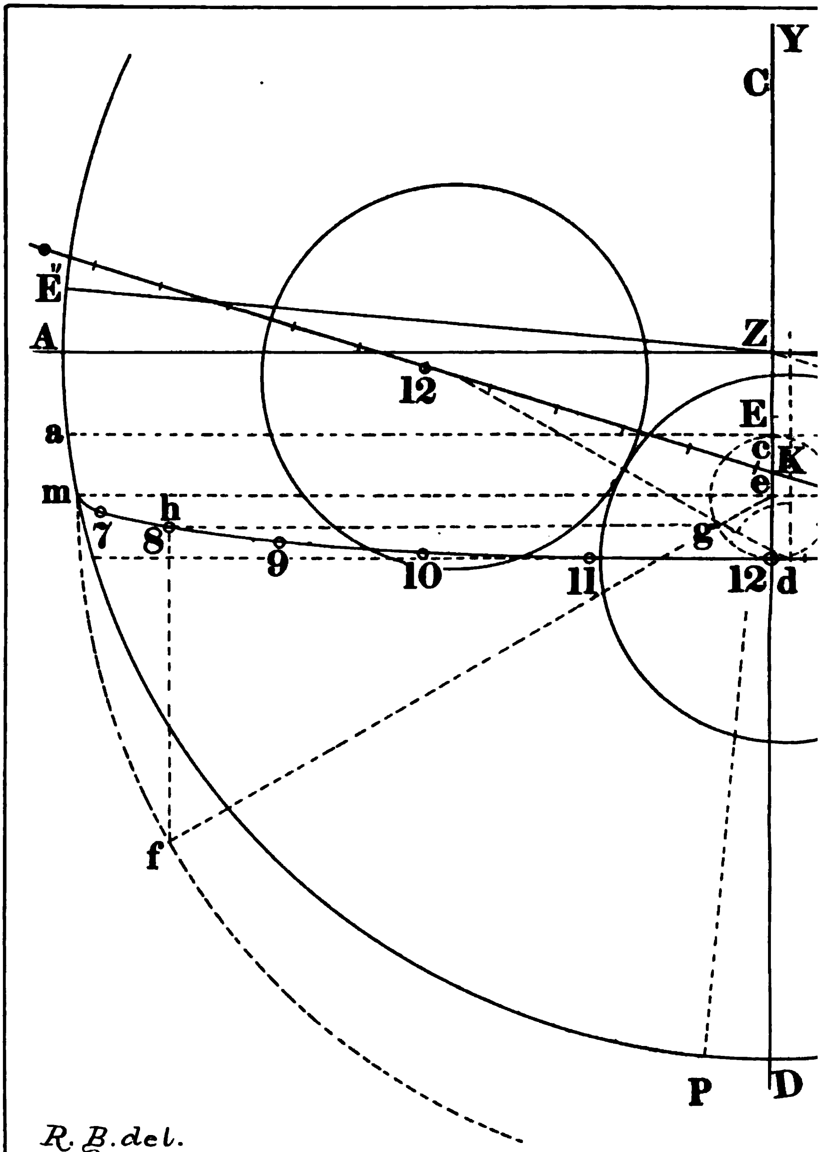

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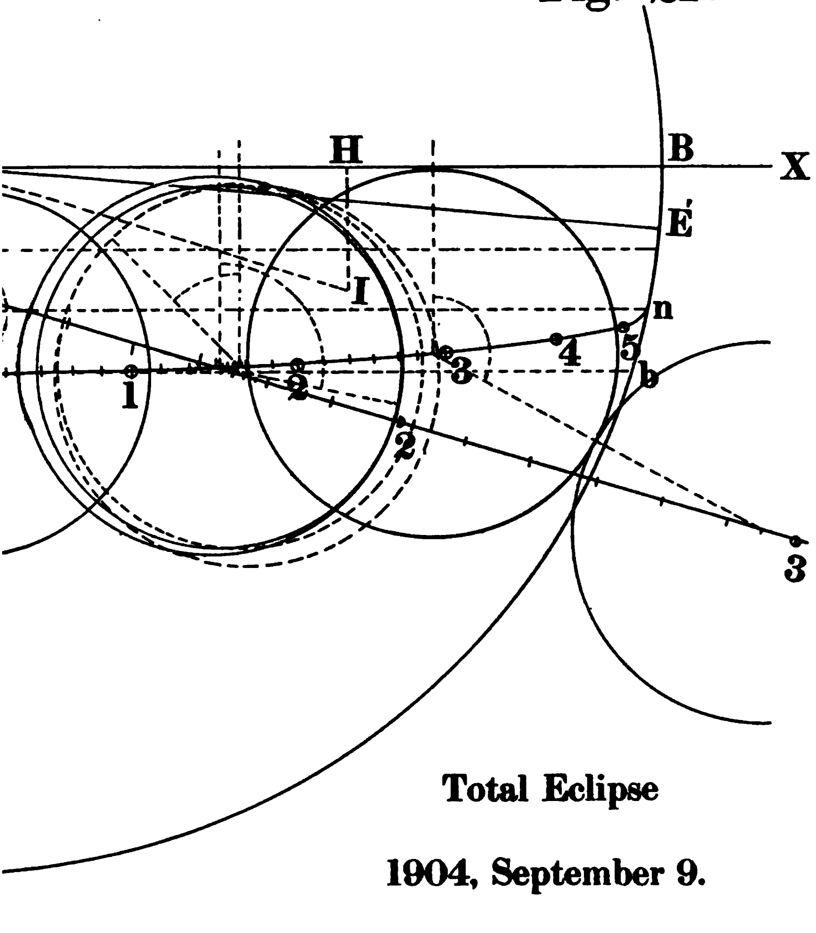
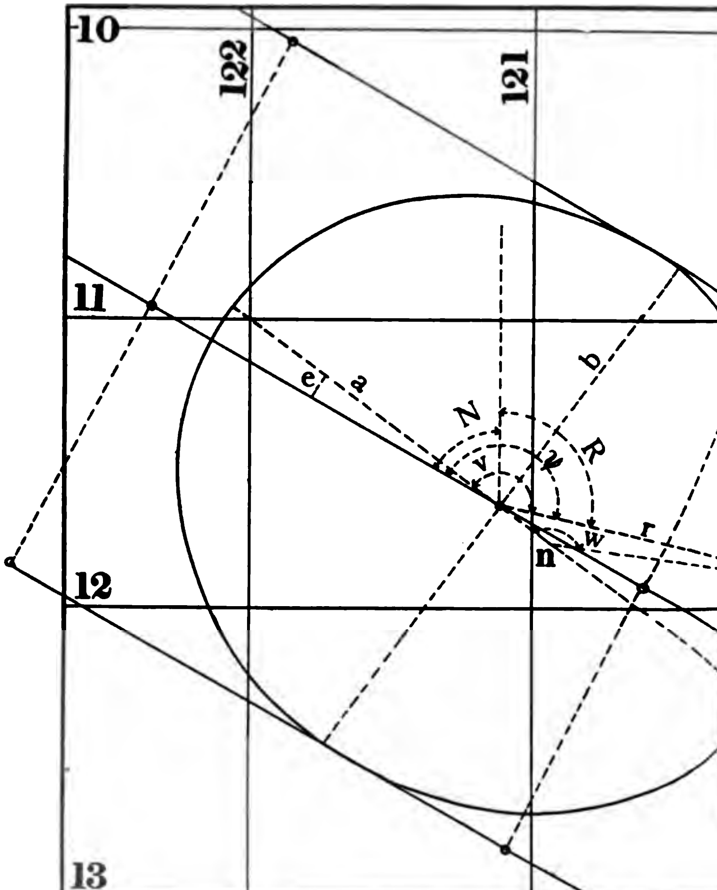
Plate VIII

Fig. 21.



Total Eclipse

1904, September 9.



Total Eclipse

1904, September 9.

Shape of the Shadow upon the Earth.

R. B. del.

SCIENTIFIC AND TECHNICAL

Print Positions

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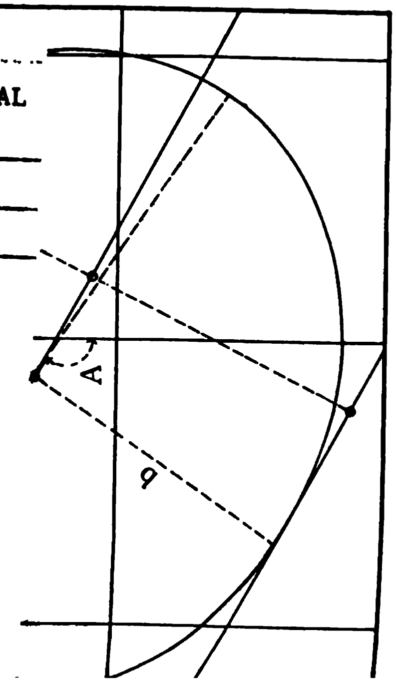
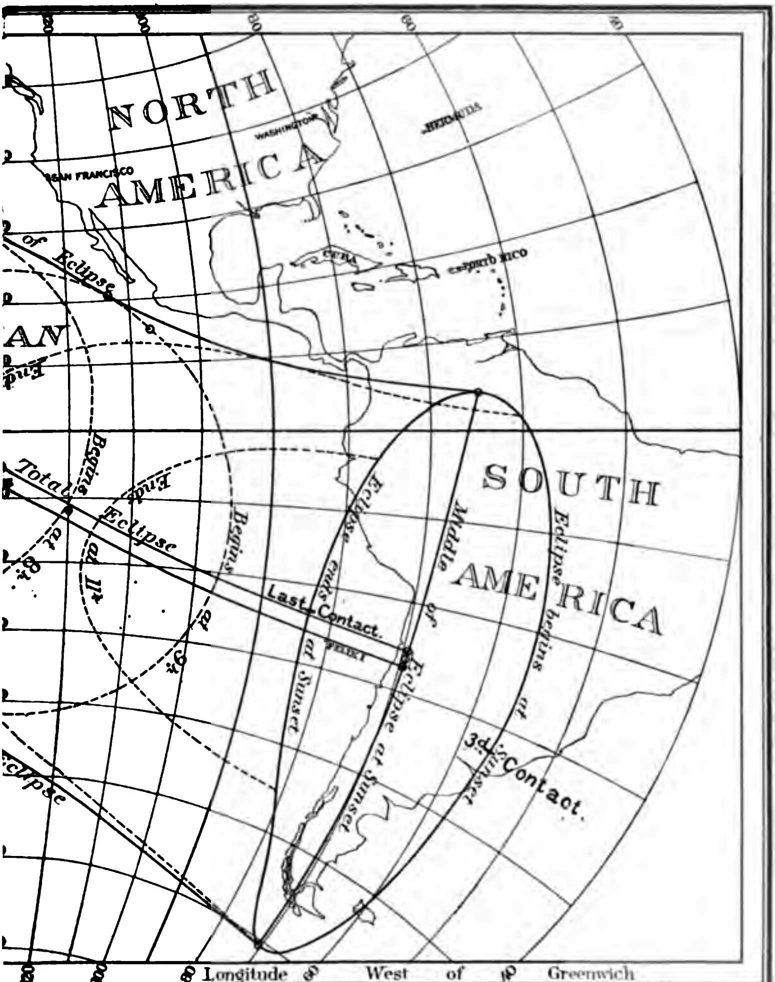
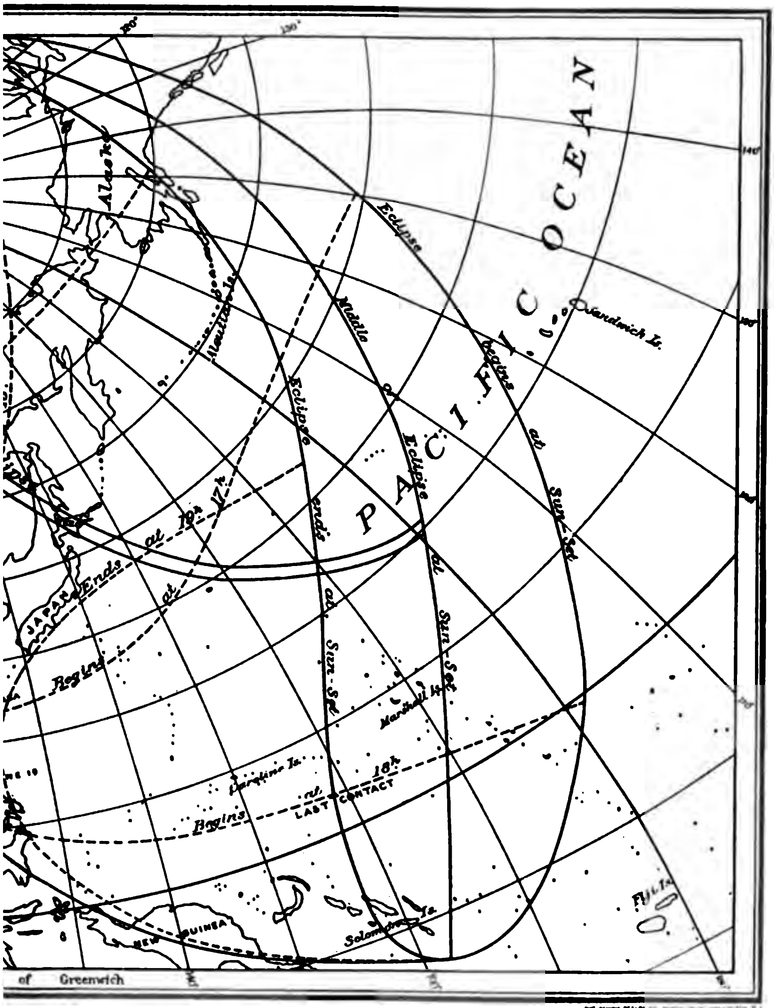


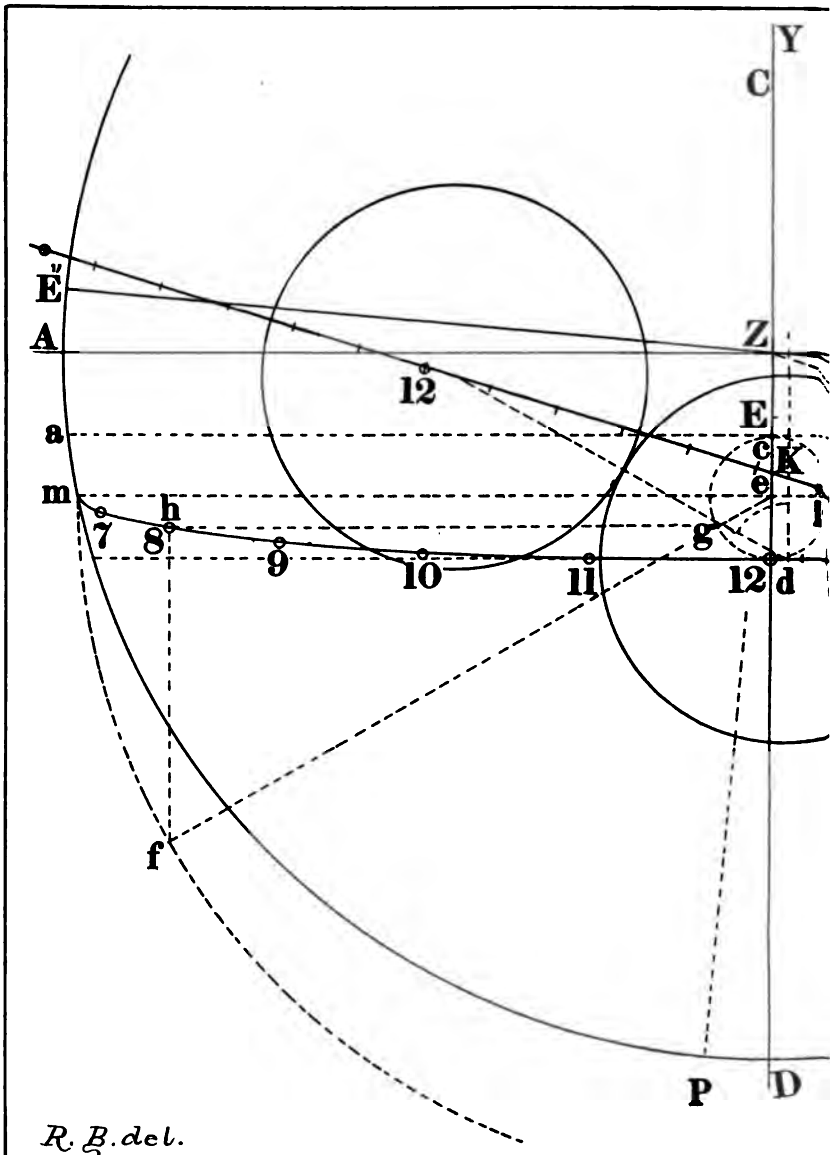
EPTEMBER 9TH 1904 **Plate VI, Fig. 17.**



are expressed in Greenwich Mean Time.

OF AUGUST 8TH 1896. Plate VII, Fig. 18.





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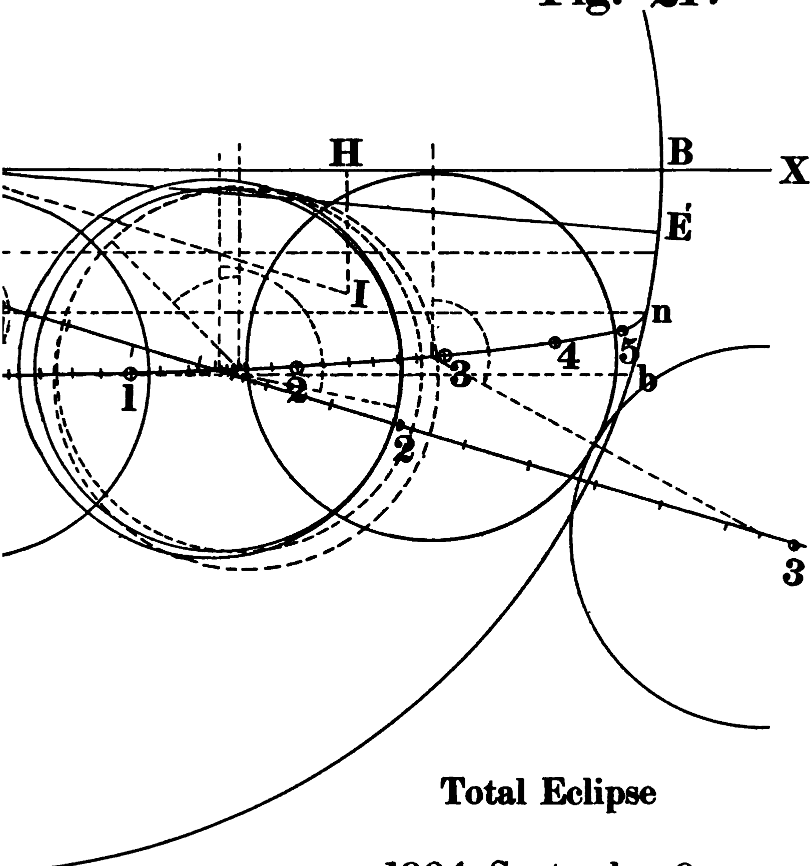
R. B. del.

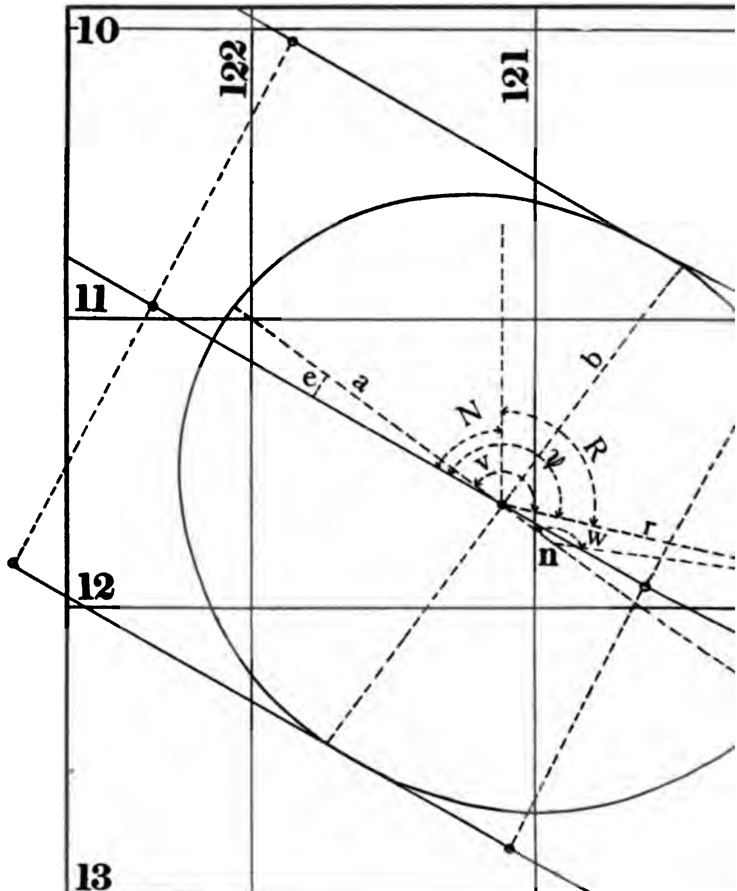
Plate VIII

Fig. 21.



Total Eclipse

1904, September 9.



Total Eclipse

1904, September 9.

Shape of the Shadow upon the Earth.

R. B. del.

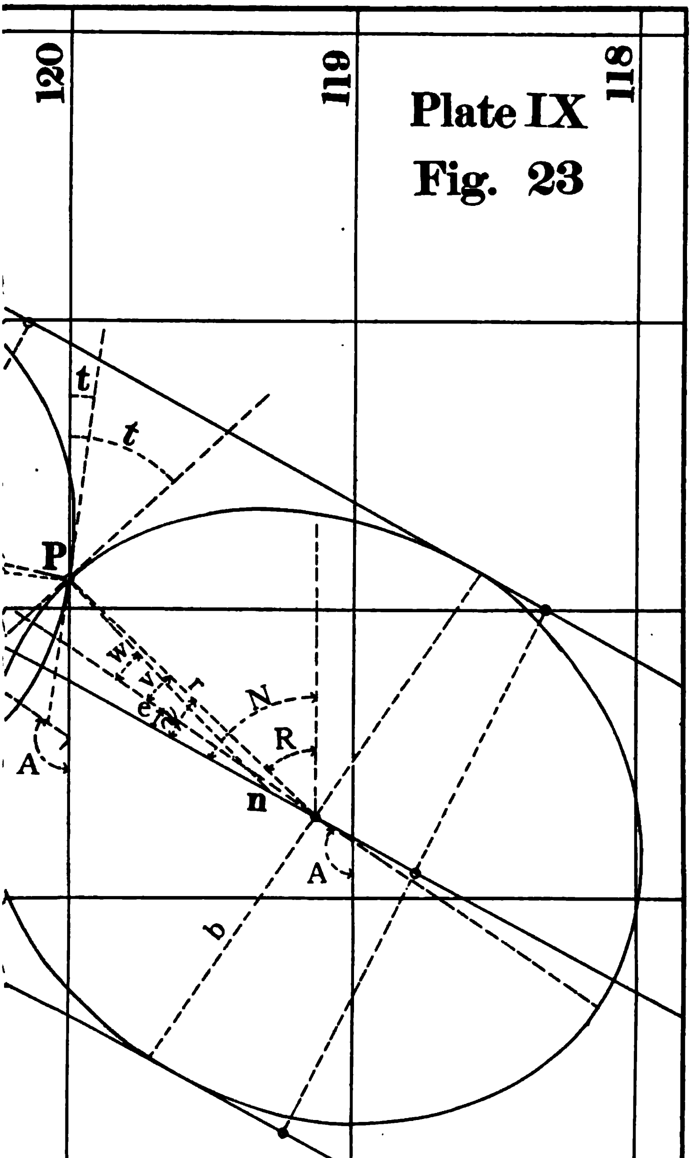
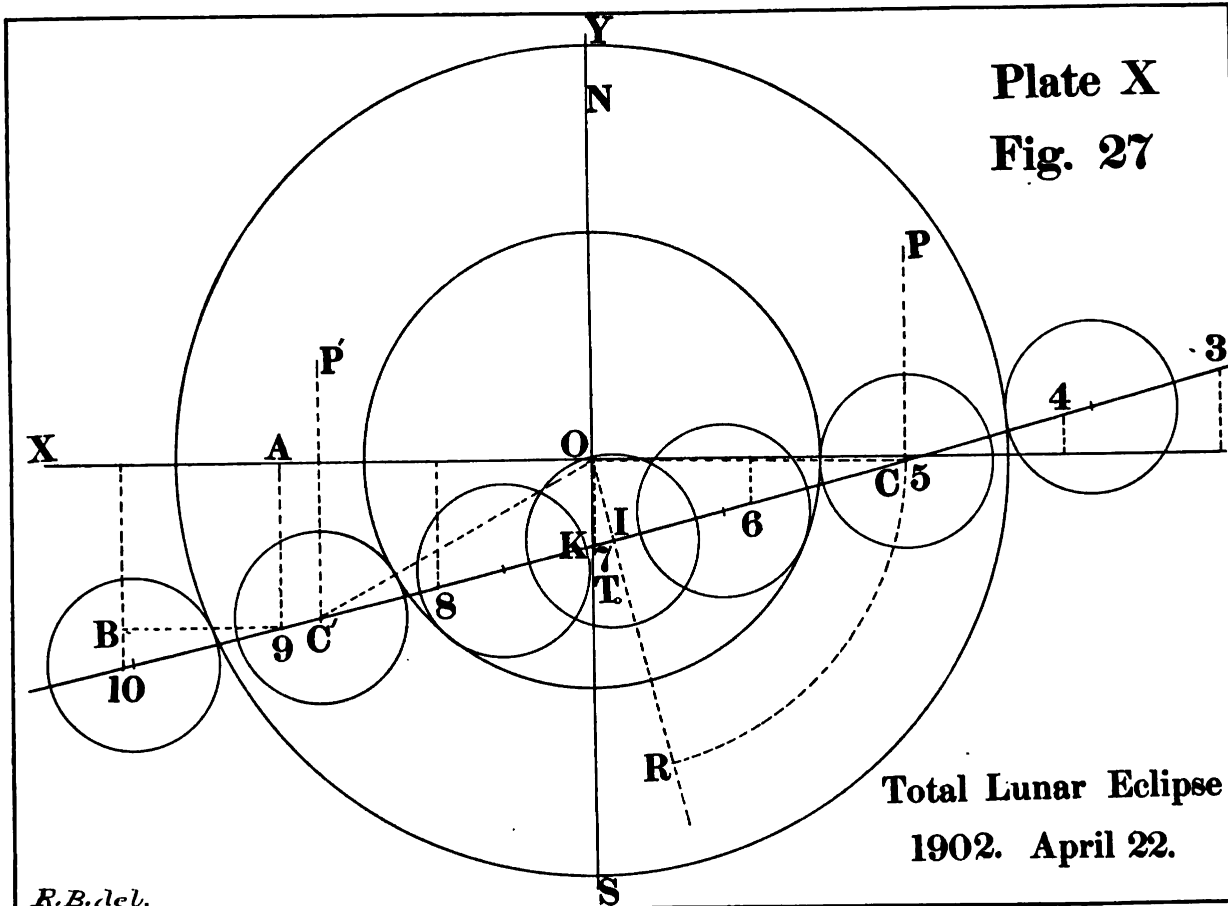


Plate IX

Fig. 23

Plate X

Fig. 27



Total Lunar Eclipse

1902. April 22.